Solutions: 3/19/18

§8.5: Power Series

(Created by Faan Tone Liu)

Key Points:

A. What is a **power series**?

- First Perspective: Inspired by polynomials, we create an "infinite-degree polynomial." For example:

$$1+x+x^2+x^3+\cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x$$

← We'll show this rigorously soon, but based on the project we did on ~~Tuesday~~ Wednesday this should seem reasonable

- Second Perspective: Put a x^n as part of a series. For example:

Centered at $x=0$: $\sum_{n=0}^{\infty} c_n x^n$

centered at $x=a$: $\sum_{n=0}^{\infty} c_n (x-a)^n$

- Third Perspective: A power series is a function where x is the input and the output is a series. For example:

$$f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$f(0)=1+\frac{0}{1!}+\frac{0^2}{2!}+\frac{0^3}{3!}+\cdots=1$$

$$f(1)=\sum_{n=0}^{\infty}\frac{(1)^n}{n!}=e^1$$

$$f(-1)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}=e^{-1}=\frac{1}{e}$$

> we'll show this precisely later. See ↑

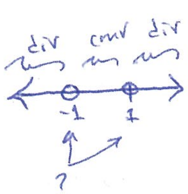
A. Basic questions:

- For what x-values dos the power series converge? To answer this question, use the <u>Ratio Test</u>. The result is an interval called the **interval of convergence**. Important: Check the endpoints separately.
- To what value does the series converge?

Examples:

1. Consider the series $1+x+x^2+\cdots+x^n+\cdots$. For which values of x does the series converge?

This is a geometric series with $a=1$, $r=x$, so for $|x|<1$, the series converges, and otherwise, the series diverges. We can also show this with the ratio test:



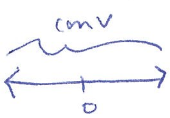
Need to check endpts:

$$1+x+x^2+\cdots+x^n+\cdots=\sum_{n=0}^{\infty}x^n \qquad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right|=\lim_{x\to\infty}|x|=|x|$$

- We need $|x|<1$ for the series to converge (by ratio test), so the interval of convergence is at least $(-1,1)$.
- We know if $|x|>1$, the series diverges (by ratio test), so we know what happens for all x except $|x|=1 \Rightarrow x=\pm 1$.
- <u>Endpts</u>: <u>$x=-1$</u>: $\sum_{n=0}^{\infty}(-1)^n$ diverges by divergence test.
  <u>$x=1$</u>: $\sum_{n=0}^{\infty}(1)^n$ diverges by div. test.

$\boxed{\text{Int of conv. is } (-1,1)}$

2. Find the interval of convergence of the series $1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$. $= \sum_{n=0}^{\infty} \frac{x^n}{n!}$



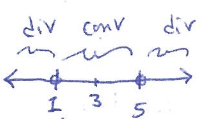
Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0.$

Since $0 < 1$, this series converges (absolutely) for all values of x.

Interval of convergence: $\boxed{(-\infty, \infty) \text{ or } \mathbb{R}}$

3. Find the interval of convergence of the series $1 - \frac{(x-3)}{2} + \frac{(x-3)^2}{4} - \frac{(x-3)^3}{8} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{2^n}$

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x-3)^n}\right| = \lim_{n\to\infty} \left|\frac{x-3}{2}\right| = \left|\frac{x-3}{2}\right|.$



?? need to check endpts

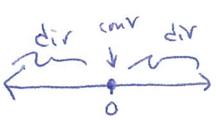
- We have absolute convergence when $\left|\frac{x-3}{2}\right| < 1 \Rightarrow -1 < \frac{x-3}{2} < 1 \Rightarrow -2 < x-3 < 2$ $\Rightarrow 1 < x < 5$

- Endpts: $x=1$: $\sum_{n=0}^{\infty} \frac{(-1)^n(-2)^n}{2^n} = \sum_{n=0}^{\infty} 1$ diverges (div. test)

  $x=5$: $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n$ diverges (div. test)

$\boxed{\text{Int of conv.} \quad (1, 5)}$

4. Find the interval of convergence of the series $\sum_{n=0}^{\infty} n!x^n$.



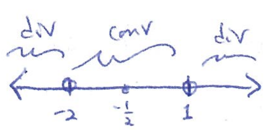
Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\,x^{n+1}}{n!\,x}\right| = \lim_{n\to\infty} |nx| = \begin{cases} \infty & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

The series diverges for all $x \neq 0$, and converges for $x = 0$.

$\boxed{\text{Interval of conv.}: \quad x = 0 \text{ or } \{0\}}$

5. Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(2x+1)^n}{n3^n}$.

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2x+1)^{n+1}}{(n+1)\,3^{n+1}} \cdot \frac{n\,3^n}{(2x+1)^n}\right| = \lim_{n\to\infty} \left|\frac{2x+1}{3} \cdot \frac{n}{n+1}\right| = \left|\frac{2x+1}{3}\right|$



?? Need to check endpts.

Series conv. abs. for $\left|\frac{2x+1}{3}\right| < 1 \Rightarrow -1 < \frac{2x+1}{3} < 1 \Rightarrow -3 < 2x+1 < 3$

$\Rightarrow -4 < 2x < 2 \Rightarrow -2 < x < 1.$

Endpoints: $x=-2$: $\sum_{n=0}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ conv. (Alt. series test)

$x=1$: $\sum_{n=0}^{\infty} \frac{3^n}{n3^n} = \sum_{n=0}^{\infty} \frac{1}{n}$ div. (p-series $p \leq 1$)

$\boxed{\text{Int. of conv:} \quad [-2, 1)}$