Solutions 3/10/18

§8.4: Absolute Convergence and Ratio Test

(Thanks to Faan Tone Liu)

Key Points:

- If the series $\sum |a_n|$ converges, then the original series $\sum a_n$ converges:

  Proof idea: $|a_1 + a_2 + \cdots + a_n + \cdots| \leq |a_1| + |a_2| + \cdots + |a_n| + \cdots$ i.e $\Sigma |a_n|$ bigger than Σa_n

  In this case, we say the series $\sum a_n$ is **absolutely convergent** or **converges absolutely**. This is a new way to prove series convergence!

- Series that are convergent but are not absolutely convergent are called **conditionally convergent**. One example is

  $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n+5}}, \quad \text{etc} \ldots$$

- The **Ratio Test** is another tool we can use. Calculate $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

  (i) If $L < 1$, then $\sum a_n$ converges absolutely (and thus converges)

  (ii) If $L > 1$, including $+\infty$, then $\sum a_n$ diverges

  (iii) If $L = 1$, the ratio test tells you nothing. Try something else.

- Helpful graphic:

	$\sum a_n$ converges	$\sum a_n$ diverges
$\sum \lvert a_n \rvert$ converges	Σa_n absolutely convergent	X not possible
$\sum \lvert a_n \rvert$ diverges	Σa_n conditionally convergent	Σa_n divergent

Examples:

1. Does $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converge or diverge? Not a positive term series, and not alternating, try something new..

Consider $\Sigma \left|\frac{\sin(n)}{n^2}\right|$: Since $0 \leq \left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$ and $\Sigma \frac{1}{n^2}$ converges (p test $p=2>1$), ~~It~~ follows that $\Sigma \left|\frac{\sin(n)}{n^2}\right|$ also converges by comparison test. Whoohoo! We know that $\Sigma \frac{\sin(n)}{n^2}$ converges absolutely.

2. Does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converge or diverge?

This series converges. Two possible proofs:

$\boxed{\text{I}}$ $\Sigma \left|\frac{(-1)^n}{n^2}\right| = \Sigma \frac{1}{n^2}$ converges by p-test $p=2>1$, so $\Sigma \frac{(-1)^n}{n^2}$ is absolutely convergent

$\boxed{\text{II}}$ $\Sigma \frac{(-1)^n}{n^2}$ is alternating. $b_n = \frac{1}{n^2}$ is decreasing & $\lim_{n\to\infty} b_n = 0$, so series converges by AST.

3. Does $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converge absolutely, converge conditionally, or diverge?

. By Alternating series test, $\sum \frac{(-1)^n}{\sqrt{n+1}}$ converges ($b_n = \frac{1}{\sqrt{n+1}}$ is decreasing + $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$.)

. However $\sum \left|\frac{(-1)^n}{\sqrt{n+1}}\right| = \sum \frac{1}{\sqrt{n+1}}$ diverges by the limit comparison test:

choose $b_n = \frac{1}{\sqrt{n}}$, $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 1$ which is positive + finite.

$\sum \frac{1}{\sqrt{n}}$ diverges by p test ($p = \frac{1}{2} \le 1$), so $\sum \frac{1}{\sqrt{n+1}}$ also diverges.

we conclude that $\sum \frac{(-1)^n}{\sqrt{n+1}}$ converges conditionally

4. Does $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converge absolutely, converge conditionally, or diverge?

Ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2} = \frac{1}{2} < 1,$$

So by the ratio test, the series converges absolutely.

5. Does $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{e^n}$ converge absolutely, converge conditionally, or diverge?

Diverges by ~~[illegible]~~ ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} = \lim_{n\to\infty} \frac{1}{e} \cdot \frac{(n+1)n!}{n!} = \lim_{n\to\infty} \frac{n+1}{e} = \infty > 1,$$

So by ratio test, the series diverges.

Note: could also use divergence test because $\lim_{n\to\infty} \frac{n!}{e^n} = \infty$, so $\lim_{n\to\infty} \frac{(-1)^n n!}{e^n}$ dne (terms don't go to 0)

6. Does $\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$ converge absolutely, converge conditionally, or diverge?

Try ratio test (factorials and exponential things)

$$\lim_{n\to\infty} \frac{(n+1)^{(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n \cdot (n+1) \cdot (2n)!}{n^n \cdot (2n+2)!}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \cdot \frac{(n+1)\,\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \cdot \frac{n+1}{4n^2+6n+2} = e \cdot 0 = 0 < 1$$

(We know $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$ and $\lim_{n\to\infty} \frac{n+1}{4n^2+6n+2} = 0$)

By ratio test, the series converges absolutely.