

1. Recall the idea of linear approximation, which we will use to approximate the numbers e^{-1} and e . Let $f(x) = e^x$. We want to find a linear function $L(x) = C_0 + C_1x$ whose value and whose derivative at $x = 0$ matches $f(x)$ at $x = 0$.

(a) Fill in the table below:

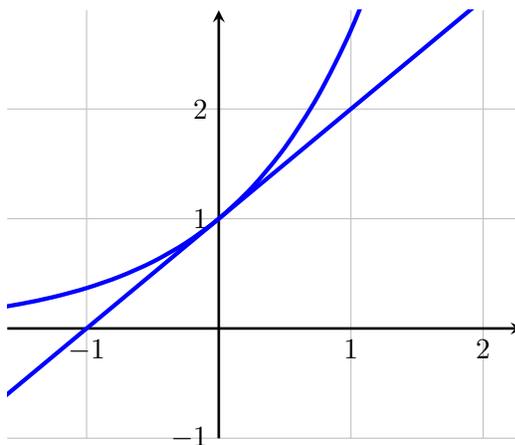
$f(x) = e^x$	$f(0) = 1$	$L(x) = C_0 + C_1x$	$L(0) = C_0$
$f'(x) = e^x$	$f'(0) = 1$	$L'(x) = C_1$	$L'(0) = C_1$

(b) Now we want to match the values and derivatives of $f(x)$ and $L(x)$, in other words:

- $f(0) = L(0)$, so $C_0 = \underline{\quad 1 \quad}$
- $f'(0) = L'(0)$, so $C_1 = \underline{\quad 1 \quad}$

(c) Thus $L(x) = 1 + x$

(d) Graph e^x and $L(x)$



- (e) Use $L(x)$ to estimate $e^{0.1}$ and e . Compare them to the estimates your calculator gives you and comment on the accuracy. Explain from the graph why the error is so much larger for estimating the value of e .

Solution: $e^{0.1} \approx L(0.1) = 1 + 0.1 = 1.1$. The calculator gives $e^{0.1} \approx 1.105\dots$. This gives us $-.006 < \text{error} < -.005$.

$e^1 \approx L(1) = 1 + 1 = 2$. The calculator gives $e^1 \approx 2.718\dots$. This gives us $-.8 < \text{error} < -.7$. The error is greater in this estimation because 1 is further from 0 than 0.1 is from 0. This is because linear estimation is centered at 0, so our estimation is exactly right at $x = 0$, and gets worse the farther we get from 0.

Note: At home you can repeat this process using an arbitrary $f(x)$ to find that the general formulas for a linear approximation centered at $x = 0$ is

$$L(x) = f(0) + f'(0)x$$

This is the equation of the tangent line, and is called the 1st degree Taylor Polynomial for $f(x)$ centered at $x = 0$.

2. Now we'll repeat the ideas of the previous problem, but using a quadratic approximation instead of a linear approximation. Let $f(x) = e^x$. We want to find a quadratic function $Q(x) = C_0 + C_1x + C_2x^2$ whose value and whose derivative and whose second derivative at $x = 0$ matches $f(x)$ at $x = 0$.

(a) Fill in the table below:

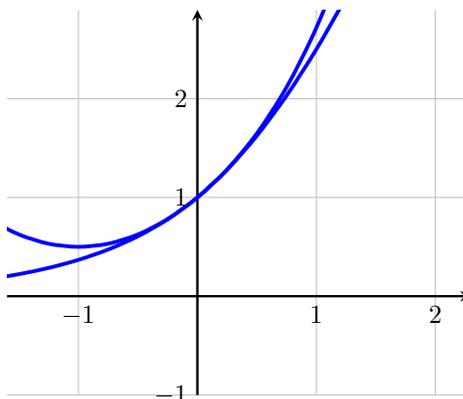
$f(x) = e^x$	$f(0) = 1$	$Q(x) = C_0 + C_1x + C_2x^2$	$Q(0) = C_0$
$f'(x) = e^x$	$f'(0) = 1$	$Q'(x) = C_1 + 2C_2x$	$Q'(0) = C_1$
$f''(x) = e^x$	$f''(0) = 1$	$Q''(x) = 2C_2$	$Q''(0) = 2C_2$

(b) Now we want to match the values and derivatives of $f(x)$ and $Q(x)$, in other words:

- $f(0) = Q(0)$, so $C_0 = \underline{1}$
- $f'(0) = Q'(0)$, so $C_1 = \underline{1}$
- $f''(0) = Q''(0)$, so $\underline{2C_2 = 1}$

(c) Thus $Q(x) = \underline{1 + x + \frac{1}{2}x^2}$

(d) A graph of e^x is shown below. Graph $Q(x)$ on the same coordinate plane.



(e) Use $Q(x)$ to estimate $e^{0.1}$ and e . Compare them to the estimates your calculator gives you and comment on the accuracy. Explain from the graph above and the graph from the previous problem why the errors are reduced.

Solution: $e^{0.1} \approx Q(0.1) = 1 + 0.1 + \frac{1}{2}(0.1)^2 = 1.105$. The calculator gives $e^{0.1} \approx 1.10517\dots$. This gives us $-0.0002 < \text{error} < -0.0001$. $e^1 \approx Q(1) = 1 + 1 + \frac{1}{2}(1)^2 = 2.5$. The calculator gives $e^1 \approx 2.718$. This gives us $-0.3 < \text{error} < -0.2$. The errors are reduced because the quadratic function is closer to the true function than the linear function at the values we estimated.

Note: At home you can repeat this process using an arbitrary $f(x)$ to find that the general formulas for a quadratic approximation centered at $x = 0$ is

$$Q(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

Notice that this is the linear approximation with an additional term added. It is called the 2nd degree Taylor Polynomial for $f(x)$ centered at $x = 0$.

3. Now we'll repeat the ideas of the previous problem, but using a cubic approximation. Let $f(x) = e^x$. We want to find a cubic function $T(x) = C_0 + C_1x + C_2x^2 + C_3x^3$ whose value and whose derivative and whose second derivative and whose third derivative at $x = 0$ matches $f(x)$ at $x = 0$.

(a) Fill in the table below:

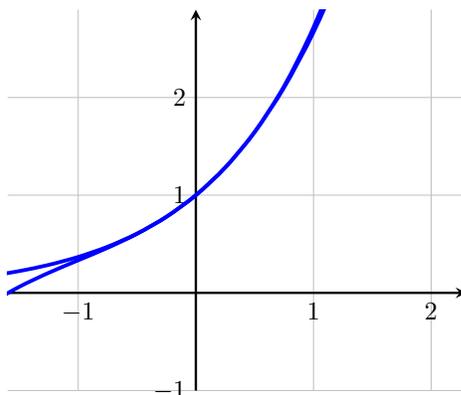
$f(x) = e^x$	$f(0) = 1$	$T(x) = C_0 + C_1x + C_2x^2 + C_3x^3$	$T(0) = C_0$
$f'(x) = e^x$	$f'(0) = 1$	$T'(x) = C_1 + 2C_2x + 3C_3x^2$	$T'(0) = C_1$
$f''(x) = e^x$	$f''(0) = 1$	$T''(x) = 2C_2 + 3 \cdot 2x$	$T''(0) = 2C_2$
$f'''(x) = e^x$	$f'''(0) = 1$	$T'''(x) = 3 \cdot 2C_3$	$T'''(0) = 6C_3$

(b) Now we want to match the values and derivatives of $f(x)$ and $T(x)$, in other words:

- $f(0) = T(0)$, so $C_0 = \underline{\quad 1 \quad}$
- $f'(0) = T'(0)$, so $\underline{\quad C_1 = 1 \quad}$
- $f''(0) = T''(0)$, so $\underline{\quad 2C_2 = 1 \quad}$
- $f'''(0) = T'''(0)$, so $\underline{\quad 6C_3 = 1 \quad}$

(c) Thus $T(x) = \underline{\quad 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \quad}$

(d) A graph of e^x is shown below. Graph $T(x)$ on the same coordinate plane.



(e) Use $T(x)$ to estimate e^{-1} and e . Compare them to the estimates your calculator gives you and comment on the accuracy. Compare the accuracy to that of the previous two problems.

Solution: $e^{0.1} \approx T(0.1) = 1 + 0.1 + \frac{1}{2}(0.1)^2 + \frac{1}{6}(0.1)^3 = 1.1051\bar{6}$. The calculator gives $e^{0.1} \approx 1.10517\dots$ This gives us $-0.000005 < \text{error} < -0.000004$. $e^1 \approx T(1) = 1 + 1 + \frac{1}{2}(1)^2 + \frac{1}{6}(1)^3 = 2.\bar{6}$. The calculator gives $e^1 \approx 2.718\dots$ This gives us $-0.06 < \text{error} < -0.05$. The accuracy has improved with every degree we added to our approximating polynomial.

Note: The general formula for a cubic approximation centered at $x = 0$ is given below. Notice that the first three terms are the same as the quadratic approximation.

$$T(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

This is called the 3rd degree Taylor Polynomial for $f(x)$ centered at $x = 0$.

4. Fill in the table below, for $f(x) = e^x$. Note that $f^{(n)}(x)$ means the n th derivative of $f(x)$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$	$\frac{f^{(n)}(0)}{n!}x^n$
0	e^x	1	$\frac{1}{0!} = 1$	1
1	e^x	1	$\frac{1}{1!} = 1$	x
2	e^x	1	$\frac{1}{2!} = \frac{1}{2}$	$\frac{1}{2}x^2$
3	e^x	1	$\frac{1}{3!} = \frac{1}{6}$	$\frac{1}{6}x^3$
4	e^x	1	$\frac{1}{4!} = \frac{1}{24}$	$\frac{1}{24}x^4$
5	e^x	1	$\frac{1}{5!} = \frac{1}{120}$	$\frac{1}{120}x^5$
6	e^x	1	$\frac{1}{6!} = \frac{1}{720}$	$\frac{1}{720}x^6$

5. Add the terms in the last column to get the 6th degree Taylor Polynomial for $f(x) = e^x$ centered at $x = 0$.

Solution: $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6$

6. Use the Taylor Polynomial from Problem 5 to estimate e .

Solution: $e^1 \approx 1 + 1 + \frac{1}{2}(1)^2 + \frac{1}{6}(1)^3 + \frac{1}{24}(1)^4 + \frac{1}{120}(1)^5 + \frac{1}{720}(1)^6 = 2.7180\bar{5}$

7. Write down the general formula for the n th degree Taylor polynomial for $f(x)$, centered at $x = 0$.

Solution: $\sum_{k=0}^n \frac{f^{(k)}(0)}{k!} (x - 0)^k$

8. Give an explanation of where the factorials come from.

Solution: Each coefficient in the Taylor polynomial is solved for by taking n derivatives of the polynomial, where n corresponds with the exponent of that term in the polynomial. n derivatives means n successive powers of x have been brought down, which creates the factorial.