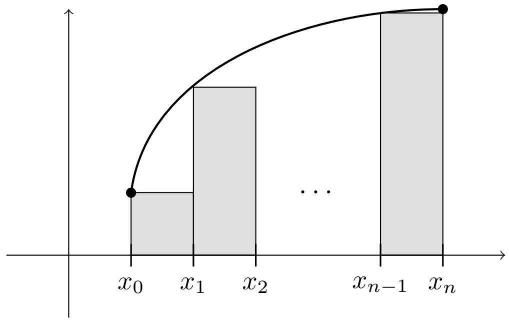
Background info, approximating $\int_a^b f(x)\,dx$.

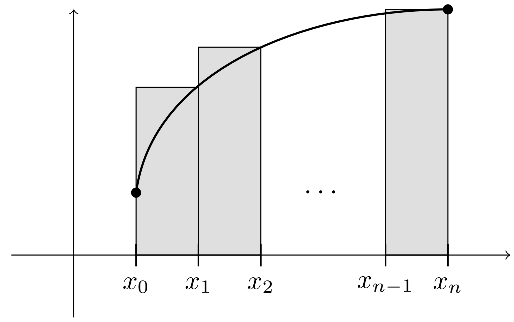
For each method, the subintervals are uniform. That is, $a = x_0$, $b = x_n$, and $\Delta x = \frac{b-a}{n}$.

Left-endpoint approximation



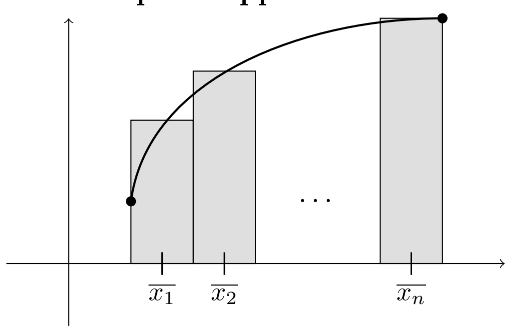
$$L_n = \Delta x[f(x_0) + f(x_1) + \cdots + f(x_{n-1})]$$

Right-endpoint approximation



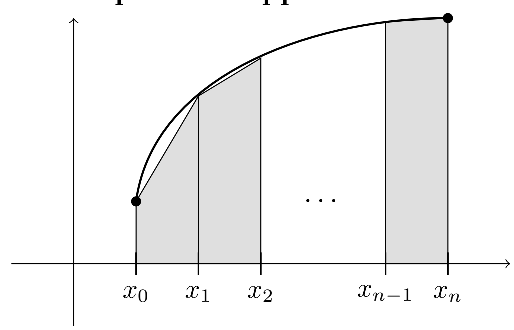
$$R_n = \Delta x[f(x_1) + f(x_2) + \cdots + f(x_n)]$$

Midpoint approximation



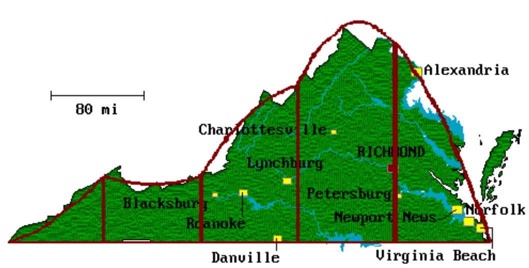
$$M_n = \Delta x[f(\overline{x_1}) + f(\overline{x_2}) + \cdots + f(\overline{x_n})]$$

Trapezoidal approximation



$$T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + f(x_n)]$$
$$T_n = \frac{1}{2}(L_n + R_n)$$

Simpson's rule (note, n must be even). Simpson's rule uses sections of parabolas to estimate areas. For more about this image see `http://www.maa.org/publications/periodicals/loci/joma/estimating-the-area-of-virginia-using-simpsons-rule`



$$S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$
$$S_{2n} = \frac{1}{3}(T_n + 2M_n)$$

1. Values of $f(x)$ are given in the table below:

x	5	7	9	11	13	15	17
$f(x)$	-2	0	1	3	4	5	8

Estimate $\int_5^{17} f(x)\,dx$ using the following methods, if possible.

With $n = 3$, $L_n =$

Solution: $4(-2 + 1 + 4) = 12$

With $n = 6$, $R_n =$

Solution: $2(0 + 1 + 3 + 4 + 5 + 8) = 42$

With $n = 6$, $T_n =$

Solution: $\frac{2}{2}(-2 + 2 \cdot 0 + 2 \cdot 1 + 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 + 8) = 32$

With $n = 6$, $M_n =$

Solution: Not possible with the given information. We don't know the values of the function at the 6 midpoints.

With $n = 3$, $M_n =$

Solution: $4(0 + 3 + 5) = 32$

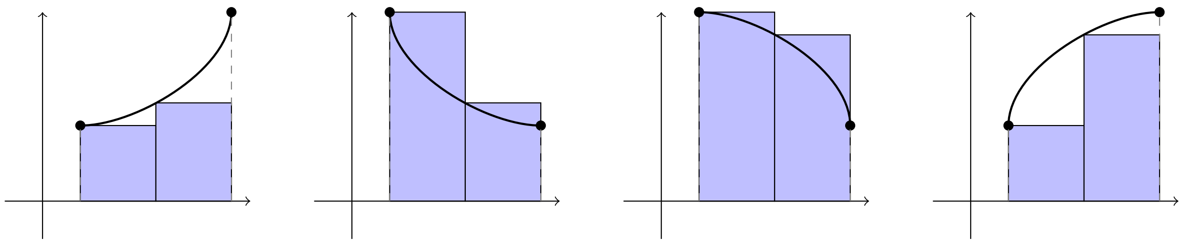
With $n = 3$, $S_n =$

Solution: Not possible because n is odd.

With $n = 6$, $S_n =$

Solution: $\frac{2}{3}(-2 + 4 \cdot 0 + 2 \cdot 1 + 4 \cdot 3 + 2 \cdot 4 + 4 \cdot 5 + 8) = 32$

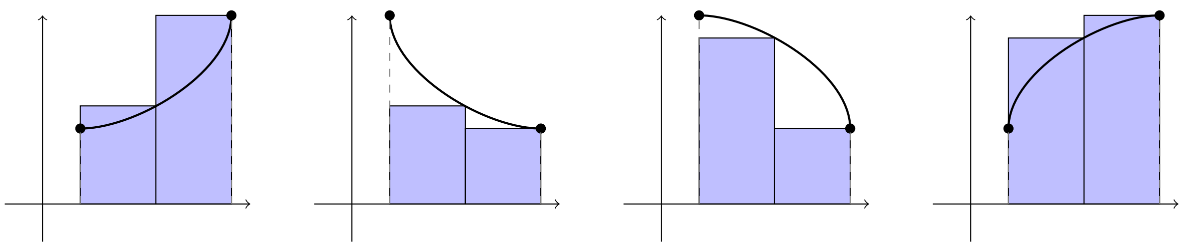
2. a. Examples of L_n. Please draw rectangles for $n = 2$.



When $f(x)$ is ________decreasing________, L_n is an overestimate.

When $f(x)$ is ________increasing________, L_n is an underestimate.

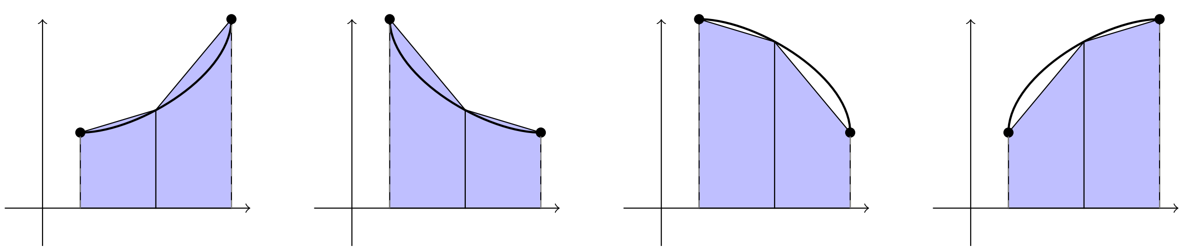
b. Examples of R_n. Please draw rectangles for $n = 2$.



When $f(x)$ is ________increasing________, R_n is an overestimate.

When $f(x)$ is ________decreasing________, R_n is an underestimate.

c. Examples of T_n. Please draw trapezoids for $n = 2$.

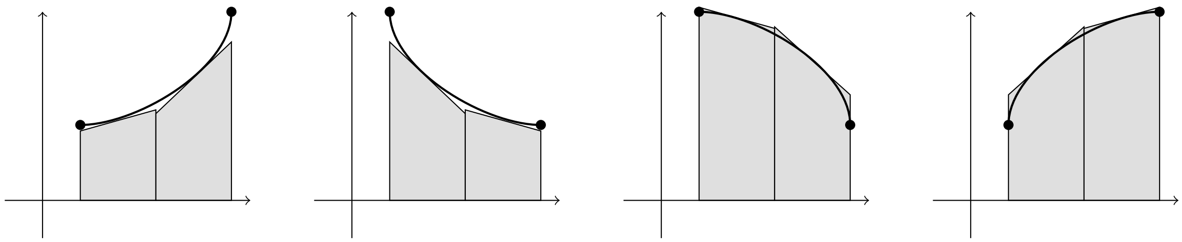


When $f(x)$ is ________concave up________, T_n is an overestimate.

When $f(x)$ is ________concave down________, T_n is an underestimate.

2. d. Examples of M_n, with $n = 2$.

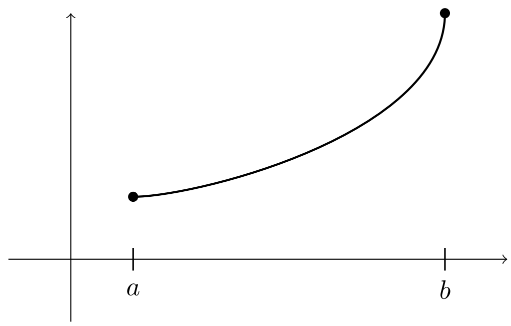
By 'rotating' the top edge of the rectangles of a Midpoint approximation, we can draw them as trapezoids.



When $f(x)$ is ____concave down____, M_n is an overestimate.

When $f(x)$ is ____concave up____, M_n is an underestimate.

3. For $f(x)$ shown below, put L_n, R_n, M_n, T_n and $\int_a^b f(x)\,dx$ in order from smallest to largest.



L_n < M_n < $\int_a^b f(x)\,dx$ < T_n < R_n

Solution: Since the graph is increasing, we know that L_n is less than $\int_a^b f(x)\,dx$ and R_n is greater than $\int_a^b f(x)\,dx$. Since the graph is concave up, we know that M_n is less than $\int_a^b f(x)\,dx$ and T_n is greater than $\int_a^b f(x)\,dx$. To see which of L_n and M_n is larger, consider the rectangles we drew to represent the areas for each of them. Since the function is increasing the rectangles for L_n are shorter than the rectangles for M_n, so $L_n < M_n$. To see which of T_n and R_n is greater, notice that since the function is increasing, the trapezoids for T_n all lie within the rectangles for R_n, so $T_n < R_n$.

Background info, error bounds (see p.405 in the textbook).

> Suppose $|f''(x)| \leq k$ for $a \leq x \leq b$. If E_T and E_M are the errors in the trapezoidal and midpoint approximations, then
>
> $$|E_T| \leq \frac{k(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \leq \frac{k(b-a)^3}{24n^2}$$

Example 1: If we use the trapezoidal approximation with $n = 10$ to estimate $\int_1^3 x^3\,dx$, how accurate are we guaranteed to be? (If you want, make a guess before you do the calculation.)

$f(x) = x^3$

$f'(x) =$ $3x^2$

$f''(x) =$ 6x

On $[1,3]$, $|f''(x)| \leq$ 18, because $f''(x)$ is increasing, max is at the right end point, $f(3) = 18$.

So, $|E_T| \leq \dfrac{18 \cdot 2^3}{12 \cdot 10^2} = .12$ (Is this more or less accurate than you guessed?)

Example 2: If we use the midpoint approximation with $n = 20$ to estimate $\int_0^1 \sin(2x)\,dx$, how accurate are we guaranteed to be?

Solution: $f(x) = \sin 2x$, $f'(x) = 2\cos 2x$, $f''(x) = -4\sin 2x$. Since $|\sin x|$ is bounded by 1, $|f''(x)| \leq 4$. This gives us the value $k = 4$.
Now using the formula for $|E_M|$, we have
$|E_M| \leq \frac{4 \cdot 1^3}{24 \cdot 20^2} = \frac{1}{2400} < .0005$. Our estimate would be within .0005.

Example 3: How large should n be to guarantee that using T_n to estimate $\int_0^1 e^{-3x}\,dx$ gives an error no larger than 0.001?

Solution: $f(x) = e^{-3x}$, $f'(x) = -3e^{-3x}$, $f''(x) = 9e^{-3x}$. $f''(x)$ is decreasing, so it is largest at the left endpoint of the interval. $f''(0) = 9$, so on the interval $[0,1]$ we have $|f''(x)| < 9$. This is our value for k. So we need $|E_T| \leq \frac{9 \cdot 1^3}{12n^2} < .001$. Solving gives $n^2 > \frac{9}{12 \cdot (.001)} = 750$. So we need $n > \sqrt{750} \approx 27.4$. We need n to be a whole number, and note that we must round up. So $n = 28$ suffices to get the desired accuracy.