Solutions:

§6.6 Part I: Work

(Created by Faan Tone Liu)

Key Points:

- $\boxed{W = \text{Force} \times \text{Distance} = F \cdot d}$

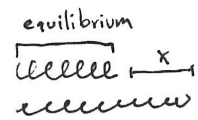
- Units:

	F = Force	d = Distance	W = Work
Metric	$N = \frac{kg\cdot m}{sec^2}$ (Newtons)	m (meters)	N·m (Newton-meters) OR J (Joules)
U.S. Units	lbs	ft	ft·lbs

- Now, what if F is not constant?

$$W = \int_a^b F(x)\,dx$$

- Dealing with springs - **Hooke's Law:**



$$F = kx\ ,$$

where x is the distance stretched or compressed past the natural (equilibrium) length, and k is the spring constant.

- Dealing with the force of gravity (metric system):

$$F = m \cdot g\ ,$$

where m is the mass of the object and $g = 9.8\frac{\text{m}}{\text{sec}^2}$.

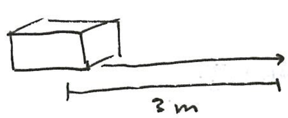
- Dealing with the force of graity (U.S. system):

$$F = \text{weight (in lbs)}.$$

Examples:

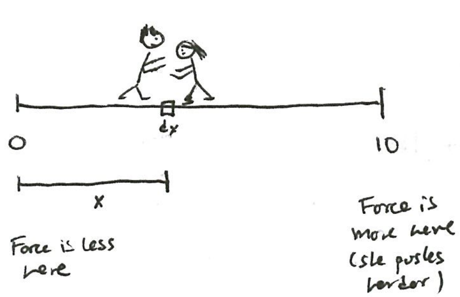
1. A box is slid 3 meters across a carpet against a force of kinetic friction of 45N. How much work is done?

Force is <u>constant</u>, so no calculus!



$$W = F \cdot d = 45N \cdot 3m = 135\ Nm$$

2. I am pushing my sister across a 10 foot room. She pushes back with increasing ferocity, with a force of $20 + \frac{x^2}{2}$ pounds, where x is how far I have pushed her. How much work do I do?

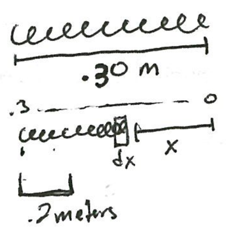


Force is <u>changing</u>!

$$\text{Work}_{slice} = F_{slice} \cdot d = \left(20 + \frac{x^2}{2}\right) \cdot dx$$

$$W = \int_0^{10} \left(20 + \frac{x^2}{2}\right) dx = \ldots = 366.\overline{6}\ \text{ft}\cdot\text{lbs}$$

3. A 30-centimeter long spring with a spring constant of $k = 120\frac{\text{N}}{\text{m}}$ is compressed to 20cm. Calculate the work done.



If spring already compressed x meters, work to compress a tiny bit (dx) more:

$$W_{slice} = F_{slice} \cdot d$$
$$= kx \cdot dx$$
$$= 120x\,dx$$

$$\text{Total Work} = \int_0^{.1} 120x\,dx = 0.6\,\text{Nm} \quad \text{OR} \quad 0.6\,\text{J}$$

4. A force of 10 lbs is required to hold a spring stretched to 6 inches past its natural length. Calculate the work required to stretch it 8 inches past its natural length.

First, find k:

$$F = kx$$
$$10\text{lbs} = k \cdot \tfrac{1}{2}\text{ft}$$
$$k = \frac{20\text{ lbs}}{\text{ft}}$$

$$W = \int_0^{2/3 \text{ ft}} 20x\,dx$$
$$= \cdots$$
$$= \frac{40}{9}\text{ ft-lbs}$$

5. How much energy is required to hoist a 3-kilogram pumpkin 15 meters to the roof of the math building?

Force is constant, so no calculus!

$$F_g = m \cdot g = 3\text{kg} \cdot 9.8 \frac{\text{m}}{\text{sec}^2} = 29.4 \frac{\text{kg m}}{\text{sec}^2} = 29.4\text{N}$$

gravity

$$W = F_g \cdot d = 29.4\text{N} \cdot 15\text{ m} = 441\text{ J}$$

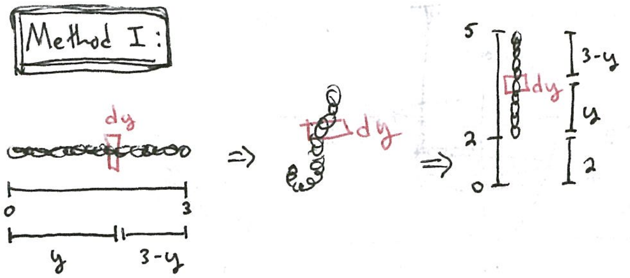
6. How much energy is required to carry a 44-lb stack of books up to the third floor of the math building? (30 ft.)

Force is constant, so no calculus!

$$W = F_g \cdot d$$
$$= 44\text{lbs} \cdot 30\text{ft} = 1320\text{ ft-lbs}$$

7. A 6-kg chain is 3 meters long. How much work is done lifting it from the ground until its lower end is 2 meters off of the ground?

Method I:



Strategy: Chop into pieces of length dy and calculate work on each:

$W_{slice} = F_g \cdot d$

$= m \cdot g \cdot d$

$= \underbrace{2\frac{kg}{m}}_{\text{mass density of chain}} \cdot \underbrace{dy}_{\text{length of slice}} \cdot \underbrace{(2+y)}_{d} = 2(2+y)\,dy$

$W_{Total} = \int_0^3 2(2+y)\,dy$

$= 205.8$ J

To find mass of slice:

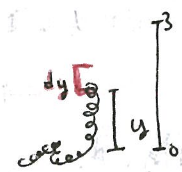
$\frac{\text{Mass whole}}{\text{Length whole}} = \frac{\text{mass slice}}{\text{Length slice}}$

$\frac{6\text{kg}}{3\text{m}} = \frac{?}{dy}$

$? = 2\frac{kg}{m}\,dy$

Method II:

Lift a large portion of the chain just a little bit (dy). Then, lift whole chain 2 m off ground



$W_{part} = F_g \cdot d$

$W_{part} = m \cdot g \cdot dy$

$= m \cdot 9.8\,dy$

To find mass of the part of chain we are lifting:

$\frac{6\text{kg}}{3\text{m}} = \frac{?}{y}$

$? = 2\frac{kg}{m} \cdot y$

$= 2y \cdot 9.8\,dy$

$= 19.6y\,dy$

Total Work

$= \underbrace{\int_0^3 19.6y\,dy}_{\text{First 3 m}} + \underbrace{F_g \cdot d}_{\text{remaining 2 m}}$

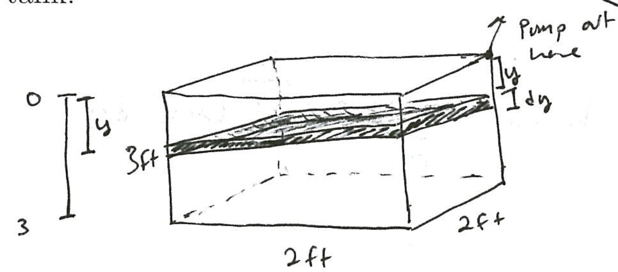
$= 88.2\text{J} + m \cdot g \cdot 2$

$= 88.2\text{J} + 6\text{kg} \cdot 9.8 \cdot$

$= 205.8$ J

8. How much work is done emptying a $2 \times 2 \times 3$-ft rectangular tank? The water must be pumped to a point in the upper corner of the tank.

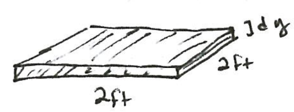
Strategy: Cut into slices of width dy and calculate work done on a slice.



$W_{slice} = F_g \cdot d$

$= \underbrace{\left(62.4\frac{\text{lbs}}{\text{ft}} \cdot 2\text{ft} \cdot 2\text{ft} \cdot dy\,\text{ft}\right)}_{F_g} \cdot \underbrace{y\,\text{ft}}_{d}$

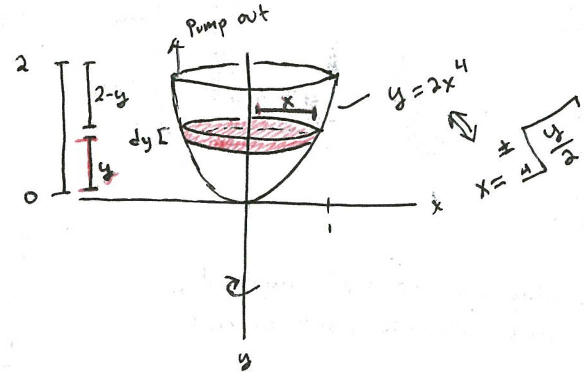
$= 249.6\,dy \cdot y$ ft-lbs



⚠ Water weighs 62.4 lbs per ft^3

Total Work $= \int_0^3 249.6y\,dy = \cdots = 1123.2$ ft lbs

9. A tub has the shape of the solid of revolution formed by rotating around the y-axis the portion of the curve $y = 2x^4$ that lies between $x = 0$ and $x = 1$. (Draw a picture.) How much work is done to empty the tank? All of the water must be pumped out of the top of the tank.



$$W_{\text{slice}} = F_g \cdot d$$

$$= m_{\text{slice}} \cdot g \cdot d$$

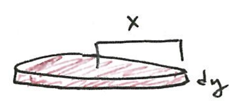
$$= \underbrace{\pi x^2 dy}_{\substack{\text{Vol. of slice} \\ (m^3)}} \cdot \underbrace{1000}_{\substack{\text{Mass density of} \\ H_2O\ (\frac{kg}{m^3})}} \cdot \underbrace{9.8}_{\substack{\text{gravity} \\ (\frac{m}{sec^2})}} \cdot \underbrace{(2-y)}_{\substack{\text{dist lifted} \\ (m)}}$$

$$= 9800\,\pi x^2 (2-y)\,dy \quad (Nm)$$

$$= 9800 \cdot \pi \cdot \sqrt{\tfrac{y}{2}}\,(2-y)\,dy \quad (J)$$

⚠ Mass density of water is

$$\frac{1g}{mL} = \frac{1000\,kg}{m^3}$$



$$\text{Vol}_{\text{slice}} = \pi x^2 \cdot dy$$

$$= \pi \left(\sqrt[4]{\tfrac{y}{2}}\right)^2 dy$$

$$= \pi \sqrt{\tfrac{y}{2}}\, dy$$

$$\text{Total Work} = \int_0^2 9800\pi \sqrt{\tfrac{y}{2}}\,(2-y)\,dy$$

$$= \cdots$$

$$\approx 32{,}840\ J \quad (\text{or } Nm)$$