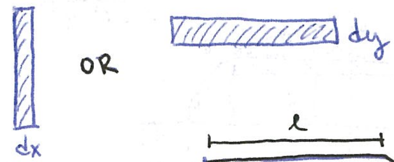
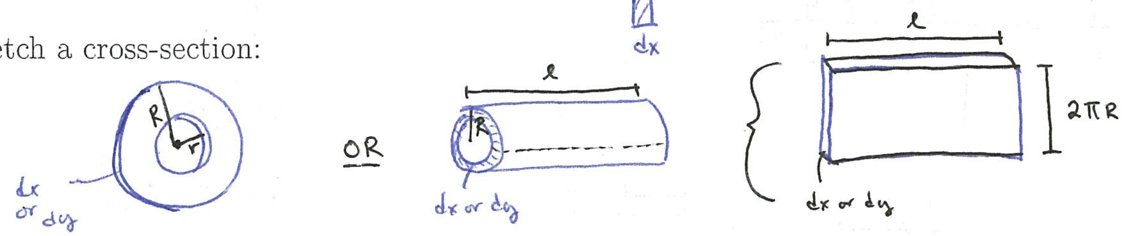
Math 2300-013: Solids of Revolution

Suggested Steps:

1. Sketch the region and axis about with the region is rotated
2. Decide on dx or dy and draw a tiny rectangle:



3. Sketch a cross-section:

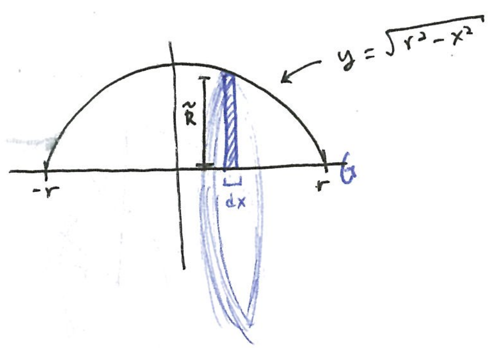


4. Find the change in volume, dV. (Substitute so all of the "pieces" are in x for a dx integral or y for a dy integral.)
5. Use an integral to sum up the volumes of the cross-sections. (Slide the rectangle you drew in Step 2 along to find the limits on your integral.)

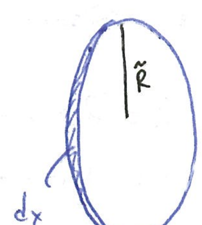
Examples:

1. In this example, we will find the volume of the sphere of radius r. Let $\mathcal{R}$ be the region bounded by the semicircle $y = \sqrt{r^2 - x^2}$ and the x-axis. Find the volume of the solid generated by rotating $\mathcal{R}$ around the x-axis.

Method I: Washers/Disks:

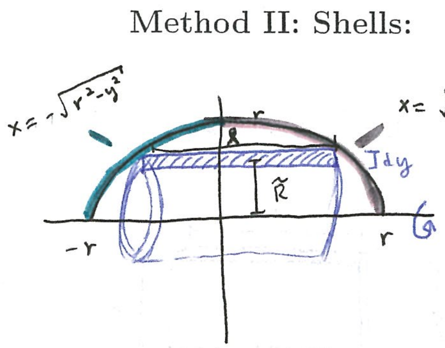


$$\text{Volume} = \int_{-r}^{r} dV$$
$$= \pi \int_{-r}^{r} r^2 - x^2\, dx$$
$$= \pi \left[r^2 x - \tfrac{1}{3}x^3 \right]_{-r}^{r}$$
$$= \pi \left[\left(r^3 - \tfrac{1}{3}r^3 \right) - \left(-r^3 + \tfrac{1}{3}r^3 \right) \right]$$
$$= \pi \left[2r^3 - \tfrac{2}{3}r^3 \right]$$
$$= \pi \left[\tfrac{4}{3} r^3 \right]$$
$$= \tfrac{4}{3}\pi r^3$$



$$dV = \pi \tilde{R}^2\, dx$$
$$= \pi \left(\sqrt{r^2-x^2}\right)^2 dx$$
$$= \pi \left(r^2 - x^2\right) dx$$

Method II: Shells:



$$\text{Volume} = \int_{y=0}^{y=r} 4\pi y\sqrt{r^2-y^2}\,dy$$

$u = r^2-y^2$

$du = -2y\,dy$

$-du = 2y\,dy$

$$= -\int_{u=r^2}^{0} 2\pi\sqrt{u}\,du$$

$$= 2\pi\int_0^{r^2} u^{1/2}\,du$$

$$= 2\pi\cdot\tfrac{2}{3}u^{3/2}\Big|_0^{r^2}$$

$$= \tfrac{4}{3}\pi r^3 - 0$$

$$= \tfrac{4}{3}\pi r^3$$

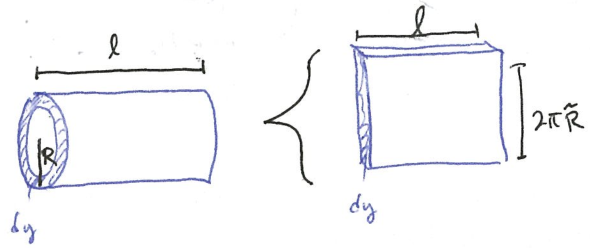
Convert eqns to $x=f(y)$:

$y=\sqrt{r^2-x^2}$

$y^2 = r^2-x^2$

$x^2 = r^2-y^2$

$x = \pm\sqrt{r^2-y^2}$



$dV = 2\pi\tilde{R}\cdot l\cdot dy$

$= 2\pi y\cdot 2\sqrt{r^2-y^2}\,dy$

$= 4\pi y\sqrt{r^2-y^2}\,dy$

$l = \sqrt{r^2-y^2} - -\sqrt{r^2-y^2}$

$l = 2\sqrt{r^2-y^2}$

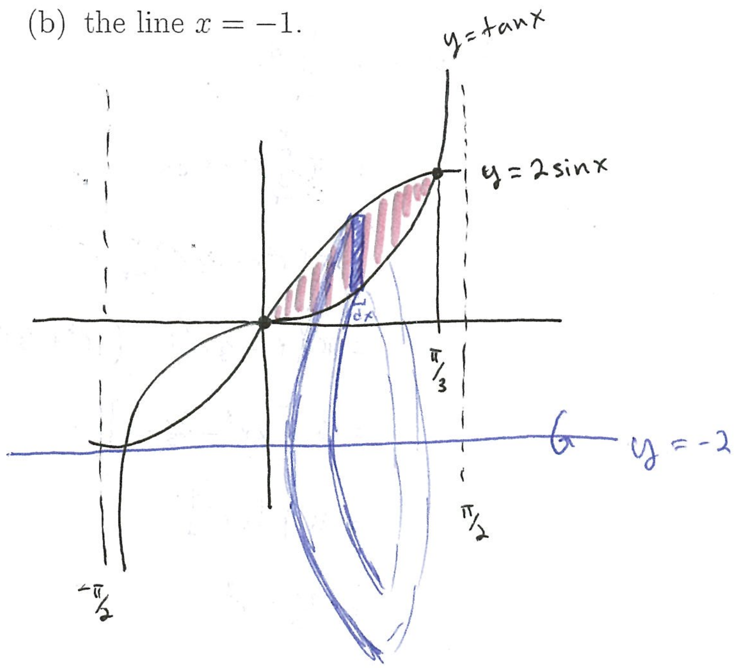
$\tilde{R} = y$

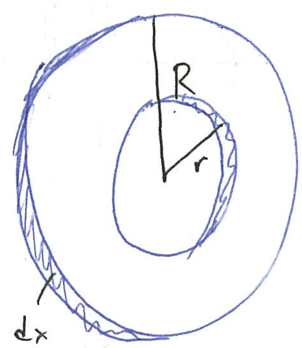
2. Let $\mathcal{R}$ be the region bounded by the curves $y = 2\sin x$ and $y=\tan x$ in the interval $[0,\pi/2)$. Find the volume of the solid formed by rotating $\mathcal{R}$ around

   (a) the line $y=-2$;

   (b) the line $x=-1$.

(a)





$R = 2\sin x - -2 = 2\sin x + 2$

$r = \tan x - -2 = \tan x + 2$

$dV = \pi R^2\,dx - \pi r^2\,dx$

$= [\pi(2\sin x+2)^2 - \pi(\tan x+2)^2]\,dx$

$= \pi\,[(2\sin x+2)^2 - (\tan x+2)^2]\,dx$

Intersection:

$2\sin x = \tan x$

$2\sin x = \dfrac{\sin x}{\cos x}$

$2\sin x\cos x = \sin x$

$2\sin x\cos x - \sin x = 0$

$\sin x\,(2\cos x - 1) = 0$

$\sin x = 0$ OR $\cos x = 1/2$

$$\text{Volume} = \int_0^{\pi/3} \pi \left[(2\sin x + 2)^2 - (\tan x + 2)^2 \right] dx$$

$$= \pi \int_0^{\pi/3} \left[4\sin^2 x + 8\sin x + \cancel{4} - \tan^2 x - 4\tan x - \cancel{4} \right] dx$$

$$= 4\pi \int_0^{\pi/3} \sin^2 x \, dx + 8\pi \int_0^{\pi/3} \sin x \, dx - \pi \int_0^{\pi/3} \tan^2 x \, dx - 4\pi \int_0^{\pi/3} \tan x \, dx$$

$$\int_0^{\pi/3} \sin^2 x \, dx = \int_0^{\pi/3} \frac{1-\cos(2x)}{2} dx$$
$$= \int_0^{\pi/3} \tfrac{1}{2} - \tfrac{1}{2}\cos(2x) \, dx$$
$$= \left[\tfrac{1}{2}x - \tfrac{1}{4}\sin(2x) \right]_0^{\pi/3}$$
$$= \left(\tfrac{\pi}{6} - \tfrac{\sqrt{3}}{8} \right) - (0-0)$$
$$= \tfrac{\pi}{6} - \tfrac{\sqrt{3}}{8}$$

$$\int_0^{\pi/3} \sin x \, dx = -\cos x \Big|_0^{\pi/3}$$
$$= -\tfrac{1}{2} + 1$$
$$= \tfrac{1}{2}$$

$$\int_0^{\pi/3} \tan^2 x \, dx = \int_0^{\pi/3} \sec^2 x - 1 \, dx$$
$$= \tan x - x \Big|_0^{\pi/3}$$
$$= \left(\sqrt{3} - \tfrac{\pi}{3} \right) - (0-0)$$
$$= \sqrt{3} - \tfrac{\pi}{3}$$

$$\int_0^{\pi/3} \tan x \, dx = \int_0^{\pi/3} \frac{\sin x}{\cos x} dx$$

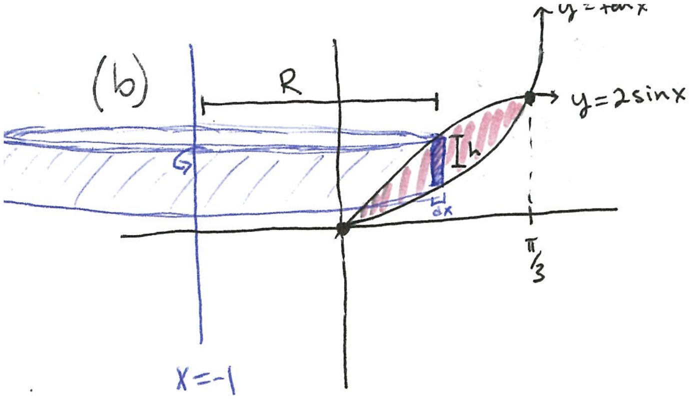
$u = \cos x$
$du = -\sin x \, dx$
$-du = \sin x \, dx$

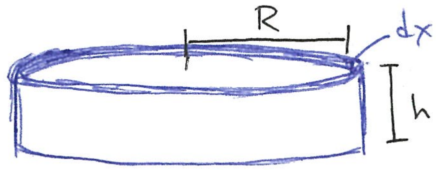
$$= -\int_1^{1/2} \frac{1}{u} du$$
$$= \int_{1/2}^{1} \frac{1}{u} du$$
$$= \ln|u| \Big|_{1/2}^{1}$$
$$= 0 - \ln(1/2)$$
$$= 0 - [\ln(1) - \ln(2)]$$
$$= \ln(2)$$

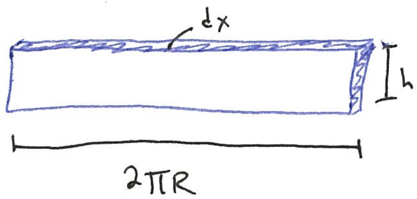
Now...

$$\text{Volume} = \ldots$$
$$= 4\pi \left(\tfrac{\pi}{6} - \tfrac{\sqrt{3}}{8} \right) + 8\pi \left(\tfrac{1}{2} \right) - \pi \left(\sqrt{3} - \tfrac{\pi}{3} \right) - 4\pi (\ln(2))$$
$$= \pi \left[\tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2} + 4 - \sqrt{3} + \tfrac{\pi}{3} - 4\ln(2) \right]$$
$$= \pi \left[\pi - \tfrac{3\sqrt{3}}{2} + 4 - \ln(16) \right]$$
$$\approx 5.5635$$

(b)







(Not to scale)

$$dV = 2\pi R \cdot h \cdot dx$$

$$dV = 2\pi(x+1)\cdot(2\sin x - \tan x)\cdot dx$$

$$\text{Volume} = \int_0^{\pi/3} 2\pi(x+1)(2\sin x - \tan x)\,dx$$

$$= 2\pi \int_0^{\pi/3} 2x\sin x + 2\sin x - (x+1)\tan x \, dx$$

$$= 4\pi \int_0^{\pi/3} x\sin x\,dx + 4\pi \int_0^{\pi/3} \sin x\,dx - 2\pi \int_0^{\pi/3} (x+1)\tan x\,dx$$

$$\approx 4\pi\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) + 4\pi\left(\frac{1}{2}\right) - 2\pi(1.20062)$$

$$\approx 9.3856$$

$$\int_0^{\pi/3} x\sin x\,dx = -x\cos x\Big|_0^{\pi/3} + \int_0^{\pi/3}\cos x\,dx$$

$u = x \quad dv = \sin x$

$du = dx \quad v = -\cos$

$$= \left(-\frac{\pi}{3}\cdot\frac{1}{2} + 0\right) + \sin x\Big|_0^{\pi/3}$$

$$= \frac{-\pi}{6} + \left(\frac{\sqrt{3}}{2} - 0\right)$$

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

$$\int_0^{\pi/3} \sin x\,dx = -\cos x\Big|_0^{\pi/3} = -\frac{1}{2} + 1 = \frac{1}{2}$$

$$\int_0^{\pi/3} (x+1)\tan x\,dx =$$ oops, I don't think this one has a nice antiderivative. Numerically, you get

≈ 1.20062.

I tried integration by parts and other techniques and failed. The moral of the story is that integrals are hard.

(Also, it's maybe possible to use washers instead of shells for this problem. Then, you have to integrate functions like ~~[illegible]~~ $\arctan^2(x)$ which looks equally bad...)