Math 2300-013: Quiz 5

Name: Solutions　　　　　　　　Score: ________

Collaborators:

Directions: This take-home quiz will be due at the beginning of class on Monday, October 9. You may use your notes, textbook, and colleagues from our class as resources, but your final write-up should be in your own words. If you work with collaborators from our class, please include their names on this quiz.

1. Determine if the sequence converges or diverges. If it converges, find the limit.

(a) $a_n = \left(1 + \frac{5}{n}\right)^{3n}$

$L = \lim_{n\to\infty} \left(1+\frac{5}{n}\right)^{3n}$　　form: 1^∞

$\ln(L) = \ln\left(\lim_{n\to\infty} \left(1+\frac{5}{n}\right)^{3n}\right)$　　$\ln(x)$ is continuous

$= \lim_{n\to\infty} \ln\left[\left(1+\frac{5}{n}\right)^{3n}\right]$

$= \lim_{n\to\infty} 3n \ln\left(1+\frac{5}{n}\right)$　　form: $\infty\cdot 0$

$= \lim_{n\to\infty} \dfrac{\ln\left(1+\frac{5}{n}\right)}{\frac{1}{3n}}$　　form: $\frac{0}{0}$

l'Hôpital's Rule

$\stackrel{H}{=} \lim_{x\to\infty} \dfrac{\frac{1}{1+\frac{5}{x}} \frac{d}{dx}\frac{5}{x}}{\frac{d}{dx}\frac{1}{3x}}$

$= \lim_{x\to\infty} \dfrac{\frac{1}{1+5/x}\cdot\frac{-5}{x^2}}{\frac{-1}{3x^2}}$

$= \lim_{x\to\infty} \left(\frac{1}{1+5/x}\cdot\frac{-5}{x^2}\cdot\frac{3x^2}{-1}\right)$

$= 15$

Hence, $\boxed{\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^{3n} = e^{15}}$

(b) $a_n = \dfrac{n^2\cos(n)}{1+n^2}$

Since $\lim_{n\to\infty} \frac{n^2}{1+n^2} = 1$ and $\lim_{n\to\infty} \cos(n)$ does not exist,

it follows that $\lim_{n\to\infty} \frac{n^2\cos(n)}{1+n^2}$ does not exist.

Optional Note: [If the limit were to exist, then $\cos(n)$ would have a limit because $a_n \cdot \frac{1+n^2}{n^2} = \cos(n)$. This is a contradiction.]

(c) $a_n = \sqrt[n]{3^n + 5^n}$

Method I:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5}{5}\left(3^n + 5^n\right)^{1/n}$$
$$= \lim_{n\to\infty} 5\left(\frac{3^n+5^n}{5^n}\right)^{1/n}$$
$$= \lim_{n\to\infty} 5\left(\left(\tfrac{3}{5}\right)^n + 1\right)^{1/n}$$
$$= 5\cdot(0+1)^0$$
$$= 5$$

Method II:

Let $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(3^n + 5^n\right)^{1/n}$

$$\ln(L) = \lim_{n\to\infty} \frac{1}{n}\ln(3^n+5^n)$$
$$= \lim_{x\to\infty} \frac{\ln(3^x+5^x)}{x} \quad \text{form: } \frac{\infty}{\infty}$$
$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{3^x+5^x}\cdot(3^x\ln 3 + 5^x \ln 5)}{1}$$
$$= \lim_{x\to\infty} \frac{3^x\ln 3 + 5^x \ln 5}{3^x + 5^x}$$
$$= \lim_{x\to\infty} \frac{\left(\frac{3}{5}\right)^x \ln(3) + \ln(5)}{\left(\frac{3}{5}\right)^x + 1}$$
$$= \frac{0+\ln(5)}{0+1}$$
$$= \ln(5).$$

Thus, $L = e^{\ln(5)} = 5.$

(d) $a_n = \ln(2n^2+1) - \ln(n^2+1)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{2n^2+1}{n^2+1}\right)$$
$\ln(x)$ is continuous
$$= \ln\left(\lim_{n\to\infty} \frac{2n^2+1}{n^2+1}\right)$$
$$= \ln\left(\frac{2}{1}\right)$$
$$= \ln(2).$$

2. A sequence a_n is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$.

(a) Show that $\{a_n\}$ is increasing and bounded above by 3.

Increasing:

① $a_1 = \sqrt{2}$
$a_2 = \sqrt{2+\sqrt{2}}$
$a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$

These first few are increasing

② Suppose $a_n \le a_{n+1}$. we will show that $a_{n+1} \le a_{n+2}$. Indeed,

$$a_n \le a_{n+1}$$
$$2 + a_n \le 2 + a_{n+1}$$
$$\sqrt{2+a_n} \le \sqrt{2+a_{n+1}}$$
$$a_{n+1} \le a_{n+2}.$$

Hence, by mathematical induction, $a_n \le a_{n+1}$ for all $n \ge 1$, so $\{a_n\}$ is increasing.

Bounded:

① $a_1 = \sqrt{2} \le 3$
$a_2 = \sqrt{2+\sqrt{2}} \le 3$

Those first couple are bounded above by 3

② Suppose $a_n \le 3$. We will show that $a_{n+1} \le 3$, too. Consider

$$a_n \le 3$$
$$2 + a_n \le 5$$
$$\sqrt{2+a_n} \le \sqrt{5}$$
$$a_{n+1} \le \sqrt{5} \le 3.$$

Hence, by mathematical induction, $a_n \le 3$ for all $n \ge 1$, so $\{a_n\}$ is bounded above by 3.

(b) Does $\{a_n\}$ converge or diverge? (Hint: use the Monotone Sequence Theorem)

Since $\{a_n\}$ is increasing and bounded, the sequence Converges by the Monotone Sequence Theorem.

(c) Find $\lim_{n\to\infty} a_n$. Let $L = \lim_{n\to\infty} a_n$. Then,

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n}$$

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$

$$L = \sqrt{2+L}$$
$$L^2 = 2 + L$$
$$L^2 - L - 2 = 0$$
$$(L-2)(L+1) = 0$$
$$L = 2 \text{ or } L = -1$$

Since the terms of a_n are all positive, we conclude that

$$\lim_{n\to\infty} a_n = 2.$$