Math 2300-013: Quiz 10

Name: Solutions Score: ______

1. (a) Find $T_3(x)$, the 3rd-degree Taylor Polynomial for $f(x) = \sqrt{x}$ centered at $a = 1$.

$$f(x) = x^{1/2} = \sqrt{x} \qquad f(1) = 1$$
$$f'(x) = \tfrac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \qquad f'(1) = \tfrac{1}{2}$$
$$f''(x) = -\tfrac{1}{4}x^{-3/2} = \frac{-1}{4\sqrt{x^3}} \qquad f''(1) = \tfrac{-1}{4}$$
$$f'''(x) = \tfrac{3}{8}x^{-5/2} = \frac{3}{8\sqrt{x^5}} \qquad f'''(1) = \tfrac{3}{8}$$

$$T_3(x) = \frac{f(1)}{0!}(x-1)^0 + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$T_3(x) = 1 + \tfrac{1}{2}(x-1) + \frac{-1}{8}(x-1)^2 + \frac{3}{8\cdot 6}(x-1)^3$$

$$\boxed{T_3(x) = 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \tfrac{1}{16}(x-1)^3}$$

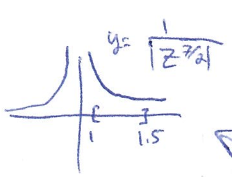
(b) If you were to approximate $\sqrt{1.5}$ using $T_3(x)$, what bound does Taylor's Inequality give you for the error?

Taylor's Inequality says that

$$|R_3(1.5)| \leq \frac{M}{\underset{\substack{\uparrow \\ n}}{(3+1)!}} |\underset{\substack{\uparrow \\ x}}{1.5} - \underset{\substack{\uparrow \\ a}}{1}|^{3+1},$$

where M is a bound on $|f^{(3+1)}(z)|$ between $x = 1.5$ and $a = 1$.
We know (from above) that

$$|f^{(4)}(z)| = \left|\frac{-15}{16} z^{-7/2}\right| = \left|\frac{-15}{16 z^{7/2}}\right| = \frac{15}{16} \cdot \left|\frac{1}{z^{7/2}}\right|$$



On the interval $1 \leq z \leq 1.5$, $\left|\frac{1}{z^{7/2}}\right|$ is biggest when $z = 1$ (because $\frac{1}{z^{7/2}}$ is decreasing). Hence, we can let $M = \frac{15}{16} \cdot 1 = \frac{15}{16}$.

We conclude that the error is $\leq \frac{15}{16 \cdot 4!}\left(\frac{1}{2}\right)^4$.

2. A function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = y^4 - 6y^3 + 5y^2.$$

(a) What are the constant solutions of the equation?

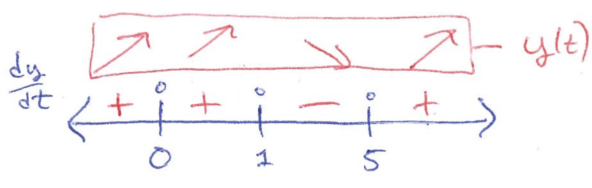
The constant solutions are those for which $\frac{dy}{dt} = 0$.
(The derivative of $y = C$ is $\frac{dy}{dt} = 0$.)

$$0 = y^4 - 6y^3 + 5y^2 = y^2(y^2 - 6y + 5) = y^2(y-1)(y-5).$$

Hence, $\boxed{y = 0,\ y = 1,\ y = 5.}$

(b) For what values of y is $y(t)$ increasing?

$y(t)$ is increasing when $y'(t) = \frac{dy}{dt}$ is positive.
Let's make a number line for $\frac{dy}{dt} = y^2(y-1)(y-5)$:



put in test points (in $\frac{dy}{dt}$)
to find the sign (+/-) of $\frac{dy}{dt}$

We conclude that $y(t)$ is increasing when $\left[\begin{array}{l} -\infty < y < 1 \\ 5 < y < \infty \end{array}\right]$ and