§5.10: Improper Integrals

Decide whether or not the following improper integrals converge or diverge.

Type I: Integrals over infinite intervals

1. $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{T\to\infty}\int_1^T \frac{1}{x}\,dx = \lim_{T\to\infty} \ln|x|\Big|_1^T$

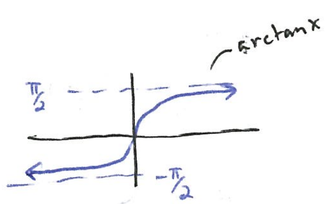
$$= \lim_{T\to\infty}(\ln T - 0)$$

$$= \infty$$

The integral diverges!

2. $\displaystyle\int_3^\infty \frac{1}{9+x^2}\,dx = \lim_{T\to\infty}\int_3^T \frac{1}{9+x^2}\,dx = \lim_{T\to\infty} \frac{1}{3}\arctan\left(\frac{x}{3}\right)\Big|_3^T$

$$= \lim_{T\to\infty}\left[\frac{1}{3}\arctan\left(\frac{T}{3}\right) - \frac{1}{3}\arctan(1)\right]$$



$$= \frac{1}{3}\cdot\frac{\pi}{2} - \frac{1}{3}\cdot\frac{\pi}{4}$$

$$= \frac{1}{3}\frac{\pi}{4} = \frac{\pi}{12}$$

⚠ Note: $\tan\left(\frac{\pi}{4}\right) = \frac{\sin(\pi/4)}{\cos(\pi/4)} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$

The integral converges!

3. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$, where $p \neq 1$

$$= \lim_{T\to\infty}\int_1^T x^{-p}\,dx$$

$$= \lim_{T\to\infty}\frac{1}{1-p}x^{1-p}\Big|_1^T$$

$$= \lim_{T\to\infty}\frac{1}{1-p}\left[T^{1-p} - 1\right]$$

Now, If $p<1$, we get $1-p>0$,

so $\displaystyle\lim_{T\to\infty} T^{1-p} = \infty$

On the other hand, if $p>1$, we get $1-p<0$,

so $\displaystyle\lim_{T\to\infty} T^{1-p} = \lim_{T\to\infty}\frac{1}{T^{p-1}} = 0$

Consequently,

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$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & \text{for } p>1 \\ \text{diverges} & \text{for } p \le 1 \end{cases}}$$

⚠ Note that this "p-rule" is for $\int_1^\infty$. →

Type II: Integrals of functions with vertical asymptotes

4. $\displaystyle\int_1^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{T\to 1^+} \int_T^2 \frac{1}{\sqrt{x-1}}\,dx$

$\displaystyle = \lim_{T\to 1^+} \int_T^2 (x-1)^{-1/2}\,dx$

$\displaystyle = \lim_{T\to 1^+} 2(x-1)^{1/2}\Big|_T^2$

$\displaystyle = \lim_{T\to 1^+}\left(2\sqrt{2-1} - 2\sqrt{T-1}\right) = 2-0=2$

The integral converges!

5. $\displaystyle\int_0^1 \ln(x)\,dx = \lim_{T\to 0^+} \int_T^1 \ln(x)\,dx$

see int. by parts worksheet

$\displaystyle = \lim_{T\to 0^+} \Big[x\ln x - x\Big]_T^1$

$\displaystyle = \lim_{T\to 0^+} \Big[(1\cdot 0 - 1) - T\ln(T) - T\Big]$

$\displaystyle = -1 - \lim_{T\to 0^+} T\ln(T)$

$= -1 - 0$

$= -1$

The integral converges!

Now,

$\displaystyle \lim_{T\to 0^+} T\ln(T) = \lim_{T\to 0^+} \frac{\ln(T)}{T^{-1}}$

L'Hôpital's Rule

$\displaystyle \overset{H}{=} \lim_{T\to 0^+} \frac{\frac{1}{T}}{-\frac{1}{T^2}}$

$\displaystyle = \lim_{T\to 0^+} -T$

$= 0$

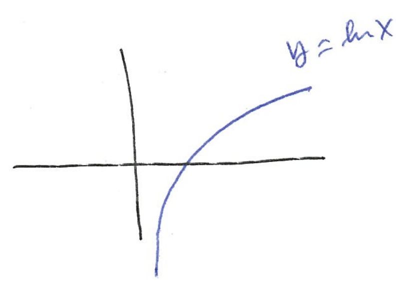
6. $\displaystyle\int_1^4 \frac{1}{x-2}\,dx$

$\displaystyle = \lim_{T\to 2^-} \int_1^T \frac{1}{x-2}\,dx + \lim_{T\to 2^+} \int_T^4 \frac{1}{x-2}\,dx$

$\displaystyle = \lim_{T\to 2^-} \ln|x-2|\Big|_1^T + \lim_{T\to 2^+} \ln|x-2|\Big|_T^4$

$\displaystyle = \lim_{T\to 2^-}\Big[\ln|T-2| - 0\Big] + \lim_{T\to 2^+}\Big[\ln|2| - \ln|T-2|\Big]$

$= -\infty + \infty$



[⚠ Note: $\infty - \infty \neq 0$]

The integral diverges!

Miscellaneous

7. $\int_{-\infty}^{\infty} te^{-t^2}\,dt$

$$= \int_{-\infty}^{0} te^{-t^2}dt + \int_{0}^{\infty} te^{-t^2}dt$$

$$= \lim_{T\to-\infty}\int_{T}^{0} te^{-t^2}dt + \lim_{T\to\infty}\int_{0}^{T} te^{-t^2}dt$$

$$\int te^{-t^2}dt = -\frac{1}{2}\int e^{u}du = -\frac{1}{2}e^{u}+C = -\frac{1}{2}e^{-t^2}+C = \frac{-1}{2e^{t^2}}+C$$

$$\left[\begin{array}{l} u=-t^2 \\ du=-2t\,dt \\ -\tfrac{1}{2}du = t\,dt \end{array}\right]$$

$$= \lim_{T\to-\infty} \left.\frac{-1}{2e^{t^2}}\right|_T^0 + \lim_{T\to\infty}\left.\frac{-1}{2e^{t^2}}\right|_0^T$$

$$= \lim_{T\to-\infty}\left[\frac{-1}{2}+\frac{1}{2e^{T^2}}\right] + \lim_{T\to\infty}\left[\frac{-1}{2e^{T^2}}+\frac{1}{2}\right]$$

$$= \frac{-1}{2}+0+0+\frac{1}{2}$$

$$= 0$$

The integral converges!

8. $\int_{1}^{\infty}\frac{1}{x\ln(x)}\,dx$

$$= \int_{1}^{2}\frac{1}{x\ln x}dx + \int_{2}^{\infty}\frac{1}{x\ln x}dx$$

$$\int \frac{1}{x\ln x}dx = \int\frac{1}{u}du = \ln|\ln x| + C$$

$$\left[\begin{array}{l} u=\ln x \\ du=\frac{1}{x}dx \end{array}\right]$$

$$= \lim_{T\to1^+}\int_{T}^{2}\frac{1}{x\ln x}dx + \lim_{T\to\infty}\int_{2}^{T}\frac{1}{x\ln x}dx$$

$$= \lim_{T\to1^+}\ln|\ln x|\Big|_T^2 + \lim_{T\to\infty}\ln|\ln|x||\Big|_2^T$$

$$= \lim_{T\to1^+}\Big[\ln|\ln(2)| - \ln|\ln(T)|\Big] + \lim_{T\to\infty}\Big[\ln|\ln(T)| - \ln|\ln(2)|\Big]$$

$$= +\infty + \infty$$

The integral diverges!

9. $\int_{-\infty}^{10}\sin^2 x\,dx$

$$= \lim_{T\to-\infty}\int_{T}^{10}\sin^2 x\,dx$$

$$= \lim_{T\to-\infty}\int_{T}^{10}\frac{1-\cos(2x)}{2}dx$$

$$= \lim_{T\to-\infty}\int_{T}^{10}\frac{1}{2}-\frac{1}{2}\cos(2x)\,dx$$

$$= \lim_{T\to-\infty}\left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right]\Big|_T^{10}$$

$$= \lim_{T\to-\infty}\left(\left[10-\frac{1}{4}\sin(20)\right] - \left[\frac{1}{2}T - \frac{1}{4}\sin(2T)\right]\right)$$

$$= \infty$$

The integral diverges!

⚠ Note: For these, we don't know what the convergent ones converge to.

Comparison Test

10. $\displaystyle\int_3^\infty \frac{\ln(x)}{\sqrt{x}}\,dx$

on $[3,\infty)$ $\quad \frac{\ln(x)}{\sqrt{x}} \geq \frac{1}{\sqrt{x}} \geq 0$

Since $\frac{1}{\sqrt{x}}$ and $\frac{\ln(x)}{\sqrt{x}}$ are continuous on $[3,\infty)$, we can use the comparison test.

$$\int_3^\infty \frac{\ln(x)}{\sqrt{x}}\,dx \geq \int_3^\infty \frac{1}{\sqrt{x}}\,dx$$

This is a divergent p-integral ($p=\frac{1}{2}\leq 1$), so since the smaller integral diverges to ∞, the larger integral also diverges.

We conclude that $\displaystyle\int_3^\infty \frac{\ln(x)}{\sqrt{x}}\,dx$ diverges.

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11. $\displaystyle\int_1^\infty \frac{|\sin x|}{x^2+1}\,dx$

On $[1,\infty)$,

$$0 \leq \frac{|\sin x|}{x^2+1} \leq \frac{1}{x^2}$$

and $\frac{|\sin x|}{x^2+1}$, $\frac{1}{x^2}$ are continuous.

Now,

$$\int_1^\infty \frac{|\sin x|}{x^2+1}\,dx \leq \int_1^\infty \frac{1}{x^2}\,dx < \infty$$

convergent p-integral ($p=2$)

Since the larger integral converges, the comparison test guarantees that the smaller integral also converges.

Hence, $\displaystyle\int_1^\infty \frac{|\sin x|}{x^2+1}\,dx$ converges.