§8.5: Power Series

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Solutions

Key Points:

A. What is a **power series**?

- First Perspective: Inspired by polynomials, we create an "infinite-degree polynomial." For example:

$$1+x+x^2+x^3+\cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \quad \leftarrow \text{we'll show this soon!}$$

- Second Perspective: Put a x^n as part of a series. For example:

Centered at $x=0$: $\sum_{n=0}^{\infty} c_n x^n$

Centered at $x=a$: $\sum_{n=0}^{\infty} c_n (x-a)^n$

- Third Perspective: A power series is a function where x is the input and the output is a series. For example:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$f(0) = \sum_{n=0}^{\infty} \frac{(0)^n}{n!} = \sum_{n=0}^{\infty} 0 = 0$$

$$f(1) = \sum_{n=0}^{\infty} \frac{(1)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e$$

$$f(-1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$$

We'll show these later!

A. Basic questions:

- For what x-values dos the power series converge? To answer this question, use the Ratio Test. The result is an interval called the **interval of convergence**. Important: Check the endpoints separately.
- To what value does the series converge?

Examples:

1. Consider the series $1 + x + x^2 + \cdots + x^n + \cdots$. For which values of x does the series converge?

- Use ratio test: $1+x+x^2+\cdots+x^n+\cdots = \sum_{n=0}^{\infty} x^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}|x| = |x|$$

Need $|x|<1$ for series to converge by ratio test. So Interval of convergence is at least $(-1,1)$.

- Check endpoints: $\underline{x=-1}$: $\sum_{n=0}^{\infty}(-1)^n$ diverges by divergence test

$\underline{x=1}$: $\sum_{n=0}^{\infty} 1$ diverges by divergence test.

- Interval of convergence is $\boxed{(-1,1)}$

2. Find the interval of convergence of the series $1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0$

Since $0<1$, this series converges (absolutely) for all values of x

Interval of convergence: $\boxed{(-\infty, \infty) \text{ or } \mathbb{R}.}$

3. Find the interval of convergence of the series $1 - \frac{(x-3)}{2} + \frac{(x-3)^2}{4} - \frac{(x-3)^3}{8} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{2^n}$

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x-3)^n}\right| = \lim_{n\to\infty} \left|\frac{x-3}{2}\right| = \frac{|x-3|}{2}$

we have absolute convergence when $\frac{|x-3|}{2} < 1 \Rightarrow -1 < \frac{x-3}{2} < 1 \Rightarrow -2 < x-3 < 2$
$\Rightarrow 1 < x < 5$.

Endpoints: $x=1$: $\sum_{n=0}^{\infty} \frac{(-1)^n (-2)^n}{2^n} = \sum_{n=0}^{\infty} 1$ diverges (div. test)

$x=5$: $\sum_{n=0}^{\infty} \frac{(-1)^n (2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n$ diverges (div. test)

$\boxed{\text{Interval of convergence}: (1,5)}$

4. Find the interval of convergence of the series $\sum_{n=0}^{\infty} n! x^n$.

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = \lim_{n\to\infty} |(n+1)x| = \begin{cases} \infty & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$

The series diverges for all $x \neq 0$, and converges for $x=0$.

Interval of convergence: $\boxed{x=0 \quad \text{OR} \quad \{0\}}$

5. Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(2x+1)^n}{n3^n}$.

Ratio Test:

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2x+1)^{n+1}}{(n+1)\, 3^{n+1}} \cdot \frac{n\, 3^n}{(2x+1)^n}\right| = \lim_{n\to\infty} \left|\frac{2x+1}{3} \cdot \frac{n}{n+1}\right| = \left|\frac{2x+1}{3}\right|$

Series abs. conv. for $\left|\frac{2x+1}{3}\right| < 1 \Rightarrow -1 < \frac{2x+1}{3} < 1 \Rightarrow -3 < 2x+1 < 3 \Rightarrow -4 < 2x < 2$
$\Rightarrow -2 < x < 1$

Endpoints: $x=-2$: $\sum_{n=0}^{\infty} \frac{(-3)^n}{n(3)^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ converges (alt series)

$x=1$: $\sum_{n=0}^{\infty} \frac{3^n}{n \cdot 3^n} = \sum_{n=0}^{\infty} \frac{1}{n}$ diverges (p-test $p \leq 1$)

$\boxed{\text{Interval of converge} \quad [-2, 1)}$