§8.2: Series

(Thanks to Faan Tone Liu)

Key Points (Part I):

- An **infinite series** is the sum of the terms of a sequence:

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$$

- More precisely, an infinite series is (related to a) special sequence of **partial sums**:

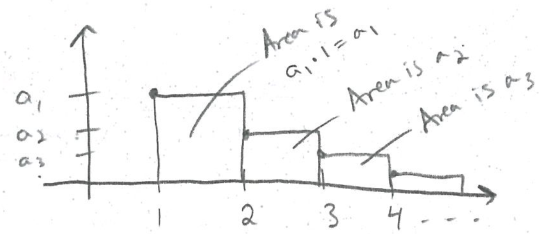
$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_N = a_1 + a_2 + \cdots + a_n = \sum_{n=1}^{N} a_n$$

- This allows us to see that the sum of an infinite series is:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_N$$

  - This is like an improper integral.
  - If $\sum_{n=1}^{\infty} a_n$ is finite, we say the series converges
  - If $\sum_{n=1}^{\infty} a_n$ has no limit (D.N.E or $\pm\infty$), we say the series diverges.

- Graphical perspective (infinite series are related to Riemann sums):



The total area of the rectangles is $\sum_{i=1}^{\infty} a_i$.

- Important tool:

Divergence Test:

If $\lim_{n\to\infty} a_n \neq 0$, then, $\sum_{n=1}^{\infty} a_n$ diverges.

⚠ Note: If $\lim_{n\to\infty} a_n = 0$, we can make no conclusion.

Examples:

1. (Using the divergence test) Does $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ converge or diverge?

$\lim_{n\to\infty} \frac{e^n}{n^2} = \infty \neq 0$, so $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ diverges by the Divergence Test

could use l'Hôpital's twice

(the terms we are adding aren't getting smaller)

2. (Harmonic series) Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge?

Try $\lim_{n\to\infty} \frac{1}{n} = 0$. Divergence test is useless!

Look @ partial sums:

$S_1 = 1 = \frac{2}{2}$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_4 = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} > \frac{4}{2}$

$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{1}{2}} > \frac{5}{2}$

⋮

Continuing the pattern, we see that the sequence of partial sums, S_N, is getting larger and larger;

$\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{N\to\infty} S_N = \infty.$

The harmonic series diverges!

3. (Telescoping Series) Explicitly calculate the sum of the series $\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$.

Partial Fractions Returns!

$\frac{1}{i(i+1)} = \frac{A}{i} + \frac{B}{i+1}$

$1 = A(i+1) + Bi$

$\underline{i=-1:}$ $1 = -B \Rightarrow B = -1$

$\underline{i=0:}$ $1 = A(1) + 0 \Rightarrow A = 1$

Thus, $\frac{1}{i(i+1)} = \frac{1}{i} + \frac{-1}{i+1}$

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{1}{i(i+1)}$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

Terms Collapse!

(Telescope Idea)

$$= \lim_{N\to\infty} \left[\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right)\right]$$

$$= \lim_{N\to\infty} \left[1 - \frac{1}{N+1}\right]$$

$$= 1$$

Key Points (PartII):

- $\sum_{n=1}^{\infty} a_n = \underline{\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n}$. (A series is a limit of a sequence of partial sums)

- The goal is to determine if $\sum_{n=1}^{\infty} a_n$ converges or diverges. So far, we have a few tools:
  - **Divergence test.** Check if $\underline{\lim_{n\to\infty} a_n = 0}$. If the limit is not zero, you are done and $\sum_{n=1}^{\infty} a_n$ __diverges__. If $a_n \to 0$, then too bad, we have to do more.
  - We can directly calculate the partial sums $S_N = \sum_{n=1}^{N} a_n$ for **telescoping series** and take the limit $\lim_{N\to\infty} S_N$ to establish convergence or divergence.
  - **Geometric series** are our friends! A geometric series has the form $\underline{\sum_{i=1}^{\infty} ar^{i-1}}$. We know that
    * $\sum_{i=1}^{n} ar^{i-1} = \underline{\frac{a-ar^n}{1-r}}$. In other words,

      $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \frac{a-ar^n}{1-r}$
    * If $|r| < 1$, then $\sum_{i=1}^{\infty} \underline{ar^{i-1} = \frac{a}{1-r}}$. In other words,

      $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \frac{a}{1-r}$
    * If $|r| \geq 1$, then $\sum_{i=1}^{\infty}$ $\underline{ar^{i-1}}$ diverges

- The **harmonic series** is $\underline{\sum_{n=1}^{\infty} \frac{1}{n}}$. It __diverges__.

- Other notes:

Examples: For each of these series, write it in expanded form if it is given in Σ-notation, and in Σ-notation if it is given in expanded form. Then, determine if the series converges and if so, find the sum.

Ex A. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{\pi}{e}\right)^{n-1} = \frac{e}{\pi} + 1 + \frac{\pi}{e} + \left(\frac{\pi}{e}\right)^2 + \cdots$

Geometric series with

$a = \text{first term} = \frac{e}{\pi}$

$r = \text{thing you multiply by} = \frac{\pi}{e}.$

Since $|r| = \left|\frac{\pi}{e}\right| \geq 1$, the series diverges. In fact, since $\lim_{n\to\infty}\left(\frac{\pi}{e}\right)^{n-1} \neq 0$, the series diverges by the Divergence test.

Ex B. $\displaystyle\sum_{i=1}^{\infty}\ln\left(\frac{i+1}{i}\right) = \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \cdots$

First try divergence test:

$\lim_{n\to\infty}\ln\left(\frac{n+1}{n}\right) = \ln(1) = 0.$

Since the limit of terms $= 0$, test fails.

Next, observe telescoping behavior...

$$\sum_{i=1}^{\infty}\ln\left(\frac{i+1}{i}\right) = \lim_{N\to\infty}\sum_{i=1}^{N}\ln\left(\frac{i+1}{i}\right) = \lim_{N\to\infty}\sum_{i=1}^{N}\left[\ln(i+1) - \ln(i)\right]$$

$$= \lim_{N\to\infty}\left[(\cancel{\ln 2} - \ln(1)) + (\ln(3) - \cancel{\ln(2)}) + \cdots + (\ln(N+1) - \ln(N))\right]$$

$$= \lim_{N\to\infty}\left[\ln(N+1) - \ln(1)\right]$$

$= \infty$. Series diverges.

Ex C. $3 + \frac{3}{2} + 1 + \frac{3}{4} + \frac{3}{5} + \frac{3}{6} + \cdots = \displaystyle\sum_{n=1}^{\infty}\frac{3}{n}$

$$= 3 \cdot \underbrace{\sum_{n=1}^{\infty}\frac{1}{n}}$$

harmonic series (diverges).

The series $\sum_{n=1}^{\infty}\frac{3}{n}$ is a constant times the harmonic series, which diverges, so $\sum_{n=1}^{\infty}\frac{3}{n}$ also diverges.

Ex D. $\displaystyle\sum_{n=2}^{\infty}3\left(\frac{2}{3}\right)^{n-1} = 2 + \frac{4}{3} + \frac{8}{9} + \cdots$

Geometric series with

$a = \text{first term} = 2$

$r = \text{thing you multiply by} = \frac{2}{3}.$

Since $|r| = \frac{2}{3} < 1$, we know that

$\sum_{n=2}^{\infty}3\left(\frac{2}{3}\right)^{n-1}$ converges to $\frac{2}{1-\frac{2}{3}} = 6$

Ex E. $5 + \sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5} + \cdots = \displaystyle\sum_{n=1}^{\infty}5^{1/n}$

Since $\lim_{n\to\infty}5^{1/n} = 1 \neq 0$,

The <u>divergence test</u> tells us that the series

$\sum_{n=1}^{\infty}5^{1/n}$ diverges.

Ex F. $1 + x + x^2 + x^3 + \cdots = \displaystyle\sum_{n=1}^{\infty}x^{n-1}$

This is a geometric series with

$a = \text{first term} = 1$

$r = \text{thing you multiply by} = x,$

so

$$\begin{cases}\text{If } |x| < 1, & \sum_{n=1}^{\infty}x^{n-1} = \frac{1}{1-x}\\ \text{If } |x| \geq 1, & \sum_{n=1}^{\infty}x^{n-1} \text{ diverges}\end{cases}$$