Background knowledge:
In the following statement, $f(x)$ is a function, $T_n(x)$ is its nth-degree Taylor polynomial centered at a, and the remainder $R_n(x) = f(x) - T_n(x)$.

Taylor's Inequality: If $f^{(n+1)}$ is continuous and $\left|f^{(n+1)}\right| \leq M$ between a and x, then:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

1. In this first example, you know the degree n of the Taylor polynomial, and the value of x, and will find a bound for how accurately the Taylor Polynomial estimates the function.

   (a) Write down the 2nd degree Taylor Polynomial for $f(x) = e^x$ centered at $a = 0$.

   Solution: $T_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{e^0}{0!}(x^0) + \frac{e^0}{1!}(x^1) + \frac{e^0}{2!}(x^2) = 1 + x + \frac{x^2}{2}$

   (b) If we want to use the Taylor Polynomial above to estimate e, what should x be?

   Solution: $e^x = e$ when $x = 1$. So x should be 1.

   (c) Use the Taylor Polynomial from part (a) to estimate e.

   Solution: $e^1 \approx T_2(1) = 1 + 1 + 1/2 = 2.5$

   (d) Find an upper bound for $f'''(x)$ for x between a and the value at which we are estimating the function, that is, between 0 and 1. This is what we call M.

   Solution: $f'''(x) = e^x$. Since e^x is increasing in x, the largest value of $f'''(x)$ for x between 0 and 1 occurs at the right-hand endpoint. Thus the maximum value is $e^1 = e$. Since $e^1 < 3$, we will use $M = 3$ as our upper bound.

   (e) Write down the error bound for $R_2(x)$, filling in values for x, a, n and M. What does this say about the accuracy of your estimate in part c?

   Solution: $|R_2(1)| = |f(1) - T_2(1)| \leq \frac{3}{(2+1)!}|1-0|^{2+1} = 0.5$

2. In this problem you'll know the value of x and the accuracy you're going for, and you will find how large a degree n for the Taylor Polynomial is needed.

   (a) Say that you want to estimate e to within 0.1. How many terms of the Taylor series do you need to add up? This time first find a bound M for $f^{(n+1)}(x)$ between a and x (notice you need to do this for arbitrary n). Then write down the error bound for $R_n(x)$, filling in values for x, a and M. Set this error bound to be less than 0.1 and solve for n.

   Solution: $f^{(n+1)}(x) = e^x$ for any n, and since e^x is increasing in x, e^1 is the largest value of $f^{(n)}(x)$ for x between 0 and 1. Since $e^1 < 3$, we will use $M = 3$ as our upper bound. Taylor's Inequality gives us

   $$|R_n(1)| \leq \frac{3}{(n+1)!}|1-0|^{n+1} = \frac{3}{(n+1)!}$$

   So we need to solve

   $$\frac{3}{(n+1)!} \leq 0.1$$

   We can't solve this analytically, but a little trial and error shows that this inequality holds for $n+1 \geq 5$, thus $n \geq 4$. So we need to use at least a 4th degree Taylor polynomial in order to guarantee an estimate within 0.1 of the true value. In fact, since $\frac{3}{5!} = .025$, our accuracy will be within .025 of the true value.

   (b) Add the number of terms you found were needed to get an estimate of e to within 0.1.

   Solution: $T_4(1) = 1 + 1 + \frac{1^2}{2} + \frac{1^3}{6} + \frac{1^4}{24} \approx 2.71$

3. In this problem you'll know the degree n of the Taylor Polynomial and the accuracy you're going for, and you will find out how large x can be. Using the 5th degree Taylor Series for $\sin x$ centered at $a = 0$ to estimate $\sin x$, how large can x be to get an estimate within .0005?

   Solution: $f^{(6)}(x) = -\sin x$ and since $|-\sin x| \leq 1$ for any x, we will use $M = 1$ as our upper bound. Taylor's Inequality gives us $|R_5(x)| \leq \frac{1}{(5+1)!}|x-0|^{5+1}$, so we need to solve $\frac{|x^6|}{6!} \leq .0005$. Solving for x gives us $|x^6| < .36$, so $-(.36)^{1/6} < x < (.36)^{1/6}$, or about $-.8434 < x < .8434$. The largest value of x that will give us an estimate of $\sin x$ that is within .0005 of the true value is $x = (.36)^{1/6}$.

4. In this problem you show that a Taylor Series for a function actually converges to the function. Show that the Taylor Series for $f(x) = \sin x$ converges to $\sin x$ for all x. This background information will be useful:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for all } x.$$

Outline of strategy:

- Get an upper bound M for $|f^{(n+1)}(x)|$ on the interval from a to x.

  Solution: Because $|f^{(n+1)}(x)|$ is either $\sin x$ or $\cos x$, $|f^{(n+1)}(x)| \leq 1$ for any x, a, or n. Therefore we will use $M = 1$ as our upper bound.

- Write down the nth degree error bound for $R_n(x)$.

  Solution: $|R_n(x)| \leq \frac{1}{(n+1)!}|x-a|^{n+1}$

- Take the limit of this bound for $R_n(x)$ as $n \to \infty$, show it is 0, for all x.

  Solution: $\lim_{n\to\infty} \frac{|x-a|^{n+1}}{(n+1)!} = 0$ (By the background info in the box above)

- State the conclusion:

  Solution: Therefore the Taylor series for $f(x) = \sin x$ centered at $a = 0$ converges, and further, as we hoped and expected, we now know that it converges to $\sin x$ for all x.

More practice:

5. (a) Find the Taylor Series directly (using the formula for Taylor Series) for $f(x) = \ln(x+1)$, centered at $a = 0$.

   Solution: (Details of the work are not shown in this answer.)

   $T(x) = \frac{\ln(0+1)}{0!}(x^0) + \frac{(0+1)^{-1}}{1!}(x^1) + \frac{-(0+1)^{-2}}{2!}(x^2) + \frac{-2(0+1)^{-3}}{3!}(x^3)\ldots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$

(b) How accurate will the estimate be if we use this series to estimate $\ln 4$ with $n = 5$?

Solution: To estimate $\ln 4$, we use $x = 3$. Taking derivatives, we find the pattern $|f^{(n+1)}(x)| = n! \cdot (x+1)^{-(n+1)}$. So $|f^{(5+1)}(x)| = 5! \cdot (x+1)^{-(5+1)}$. This is a decreasing function, so its maximum value between $a = 0$ and $x = 3$ occurs at the left-hand endpoint of the interval $[0, 3]$. We see that $|f^{(5+1)}(x)| = 5! \cdot (x+1)^{-(5+1)}$ is bounded by $5!$. So the error is bounded by $|R_5(3)| \leq \frac{5!}{6!}3^6 = 121.5$. This is not very accurate!

(c) Show that this series converges to $\ln(x+1)$ on the interval $(-\frac{1}{2}, \frac{1}{2})$. Note: do not use the ratio test, since it only shows convergence of the series, not convergence to the correct function. Instead, show that the limit of the error term is 0.

Solution: Taking derivatives, we find the pattern $|f^{(n+1)}(x)| = n! \cdot (x+1)^{-(n+1)}$. No matter what n is, this is a decreasing function, so its maximum on the interval $-\frac{1}{2} < x < \frac{1}{2}$ occurs at the left-hand endpoint, $x = -\frac{1}{2}$. This gives $M = |f^{(n+1)}(-1/2)| = 2^{n+1}n!$

So $|R_n(x)| \leq \frac{2^{n+1}n!}{(n+1)!}|x|^{n+1} = \frac{2^{n+1}}{n+1}|x|^{n+1}$.

Since $-\frac{1}{2} < x < \frac{1}{2}$, we now have $|R_n(x)| \leq \frac{2^{n+1}}{n+1}|1/2|^{n+1} = \frac{1}{n+1}$.

$\lim_{n\to\infty} \frac{1}{n+1} = 0$ and therefore the series converges to $\ln(x+1)$ on the interval $(-\frac{1}{2}, \frac{1}{2})$.

(Note: You could also show this by showing the alternating series error (that is, the next term in the series) goes to zero.)

(d) For $x = \frac{1}{4}$, what degree Taylor polynomial do we need to use to guarantee an approximation correct to within 4 decimal places (that is, to within .00005)?

Solution: $|f^{(n+1)}(x)| = n! \cdot (x+1)^{-(n+1)}$, so between $a = 0$ and $x = 1/4$,

$$|f^{(n+1)}(x)| \leq n!$$

Therefore,

$$|R_n(x)| \leq \frac{n!}{(n+1)!}|1/4|^{n+1} = \frac{1}{n+1}|1/4|^{n+1}$$

Solving $\frac{1}{n+1}|1/4|^{n+1} \leq .00005$ gives us $n \geq 5$. So we need to use a Taylor polynomial of at least the 5th degree to get an approximation correct to within 4 decimal places.

(Note: This could also be found using the alternating series error term.)

6. Show that the 6th degree Taylor Polynomial for $\cos x$, centered at 0, gives values which are accurate to at least four decimal places (to within .00005) if $|x| < 1$.

Solution: Because $|f^{(n+1)}(x)|$ is either $\sin x$ or $\cos x$, $0 \le |f^{(n+1)}(x)| \le 1$ for any x or n. Therefore we will use $M = 1$ as our upper bound. Using the fact that $|x| < 1$, we have

$$|R_6(x)| \le \frac{1}{(6+1)!}|x-0|^{6+1} \le \frac{1}{7!}|1|^7 \le .0002$$

Not good enough! I notice something very tricky: The 6th degree and 7th degree T.P. for $\cos x$ are the same! So the error in the 6th degree T.P. is the same as the error for the 7th degree. I can still use $M = 1$.

$$|R_6(x)| = |R_7(x)| \le \frac{1}{(7+1)!}|x-0|^{6+1} \le \frac{1}{8!}|1|^7 \le .0000249 < .00005$$

7. Find $\sin(35°)$ to within 3 decimal places (to within .0005). Note that you have options here about where to place the center a.

Solution: We'll estimate this using $a = 0$, but it could also be done by centering about $a = 30° = \frac{\pi}{6}$. First, we need to use radians, so we're going to approximate $\sin(7\pi/36)$. Second, we need to know what degree Taylor polynomial we should use to guarantee accuracy to within .0005. Because $|f^{(n+1)}(x)|$ is either $\sin x$ or $\cos x$, $0 \le |f^{(n+1)}(x)| \le 1$ for any x, a, or n. Therefore we will use $M = 1$ as our upper bound, regardless of what n is.
If we use $a = 0$, then

$$|R_n(7\pi/36)| \le \frac{1}{(n+1)!}|7\pi/36|^{n+1}$$

We need to solve $\frac{1}{(n+1)!}|7\pi/36|^{n+1} \le .0005$ for n. We can't solve this analytically, but trial and error shows that the answer is $n \ge 5$.

Therefore

$$T_5(7\pi/36) = (7\pi/36) - \frac{(7\pi/36)^3}{3!} + \frac{(7\pi/36)^5}{5!} \approx 0.57358$$

estimates $\sin(35°)$ to within 3 decimal places.