Worksheet Purpose: A few weeks ago we saw that a given improper integral converges if its integrand is less than the integrand of another integral known to converge. Similarly a given improper integral diverges if its integrand is greater than the integrand of another integral known to converge. In problems 1-7 you'll apply a similar strategy to determine if certain series converge or diverge. Additionally, in problems 8 and 9 you'll apply a different method (using limits) to determine if a series converges or diverges.

1. For each of the following situations, determine if $\sum_{n=1}^{\infty} c_n$ converges, diverges, or if one cannot tell without more information.

   (a) $0 \le c_n \le \frac{1}{n}$ for all n, we can conclude that $\sum c_n$ __unknown, not enough info.__

   (b) $\frac{1}{n} \le c_n$ for all n, we can conclude that $\sum c_n$ __diverges__

   (c) $0 \le c_n \le \frac{1}{n^2}$ for all n, we can conclude that $\sum c_n$ __converges__

   (d) $\frac{1}{n^2} \le c_n$ for all n, we can conclude that $\sum c_n$ __unknown, not enough info.__

   (e) $\frac{1}{n^2} \le c_n \le \frac{1}{n}$ for all n, we can conclude that $\sum c_n$ __unknown, not enough info.__

2. **Follow-up to problem 1:** For each of the cases above where you needed more information, give (i) an example of a series that converges and (ii) an example of a series that diverges, both of which satisfy the given conditions.

   Solution:

   (a) (i) $c_n = 1/n^2$, (ii) $c_n = 1/(2n)$.
   (d) (i) $c_n = 2/n^2$, (ii) $c_n = 1/n$.
   (e) (i) $c_n = 2/n^2$, (ii) $c_n = 1/(2n)$.

3. Fill in the blanks:

The Comparison Test (also known as Term-size Comparison Test or Direct Comparison Test)
Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ __converges__ and $a_n \le b_n$, then $\sum a_n$ also __converges__.
- If $\sum b_n$ __diverges__ and $a_n \ge b_n$, then $\sum a_n$ also __diverges__.

Note: in the above theorem and for the rest of this worksheet, we will use $\sum b_n$ to represent the series whose convergence/divergence we already know (p-series or geometric), and $\sum a_n$ will represent the series we are trying to determine convergence/divergence of.

Now we'll practice using the Comparison Test:

4. Let $a_n = \dfrac{1}{2^n + n}$ and let $b_n = \left(\dfrac{1}{2}\right)^n$ for $n \geq 1$, both sequences with positive terms.

   (a) Does $\displaystyle\sum_{n=1}^{\infty} b_n$ converge or diverge? Why?

   Solution: It converges, because it is a geometric series with $r = \frac{1}{2}$, $|r| < 1$

   (b) How do the size of the terms a_n and b_n compare?

   Solution: $a_n < b_n$ because a_n has a bigger denominator

   (c) What can you conclude about $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + n}$?

   Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + n}$ also converges, by the comparison test.

5. Let $a_n = \dfrac{1}{n^2 + n + 1}$, a sequence with positive terms.
   Consider the rate of growth of the denominator. This hints at a choice of:

   $b_n =$ $\underline{\quad \frac{1}{n^2} \quad}$, another positive term sequence.

   (a) Does $\sum b_n$ converge or diverge? Why?

   Solution: $\sum \frac{1}{n^2}$ converges, because it is a p-series with $p = 2 > 1$

   (b) How do the size of the terms a_n and b_n compare?

   Solution: $a_n < b_n$ because a_n has a bigger denominator

   (c) What can you conclude about $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1}$?

   Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1}$ also converges by the comparison test.

6. Use the Comparison Test to determine if $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n^4 + 1}}{n^3 - 2}$ converges or diverges.

Solution: We have $a_n = \frac{\sqrt{n^4+1}}{n^3-2}$ and we'll choose $b_n = \frac{n^2}{n^3}$. $a_n > b_n$ because a_n has a bigger numerator and a smaller denominator than b_n. $b_n = \frac{n^2}{n^3} = \frac{1}{n}$, so $\sum_{n=1}^{\infty} b_n$ diverges (p-series, with $p = 1$). Finally, by the comparison test, $\sum_{n=1}^{\infty} a_n$ also diverges.

7. Use the Comparison test to determine if $\sum_{n=1}^{\infty} \frac{\cos^2 n}{\sqrt{n^3+n}}$ converges or diverges.

Solution: $a_n = \frac{\cos^2 n}{\sqrt{n^3+n}}$, which we'll compare to $b_n = \frac{1}{n^{3/2}}$. a_n has a smaller numerator and a bigger denominator than b_n, so $a_n \le b_n$. $\sum b_n$ converges (p-series with $p = \frac{3}{2} > 1$). By the Comparison Test, $\sum a_n$ also converges.

8. Disappointingly, sometimes the Comparison Test doesn't work like we wish it would. For example, let $a_n = \frac{1}{n^2-1}$ and $b_n = \frac{1}{n^2}$ for $n \ge 2$.

   (a) By comparing the relative sizes of the terms of the two sequences, do we have enough information to determine if $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges or diverges?

   Solution: No: $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$), but we can't conclude that $\sum_{n=2}^{\infty} a_n$ does, because we can't say that $a_n \le b_n$ for all $n \ge 2$.

   (b) Show that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

   Solution:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1.$$

   (c) Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, we know that $a_n \approx b_n$ for large values of n. Do you think that $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n^2-1}$ must converge?

   Solution: Yes. The intuition is that even though a_n is slightly larger than b_n, as n gets big, a_n and b_n become essentially the same. Since $\sum_{n=2}^{\infty} b_n$ is finite, so is $\sum_{n=2}^{\infty} a_n$.

When we have chosen a good series to compare to, but the inequalities don't work in our favor, we use the Limit Comparison Test instead of the Comparison Test.

The Limit Comparison Test
Suppose $a_n > 0$ and $b_n > 0$ for all n. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is finite and $c > 0$, then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Now we'll practice using the Limit Comparison Test:

9. Determine if the series $\sum_{n=2}^{\infty} \frac{n^3 - 2n}{n^4 + 3}$ converges or diverges.

Solution: Call $a_n = \frac{n^3-2n}{n^4+3}$. Looking at the most dominant terms in the numerator and denominator, it seems like we should compare to $b_n = \frac{n^3}{n^4} = \frac{1}{n}$. We start by trying to use the Comparison Test. Our sequence a_n is smaller than b_n (because it has a smaller denominator and bigger numerator than $\frac{n^3}{n^4}$). However, $\sum \frac{1}{n}$ diverges (harmonic series). So unfortunately, the inequality is going the wrong way to give a conclusion using the Comparison test. We'll use the Limit Comparison Test instead. We find the limit of the ratio of the two sequences:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{n^3-2n}{n^4+3}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{n^3 - 2n}{n^4 + 3} \cdot \frac{n}{1} \\
&= \lim_{n\to\infty} \frac{n^4 - 2n^2}{n^4 + 3} \\
&= \lim_{n\to\infty} \frac{n^4 - 2n^2}{n^4 + 3} \cdot \frac{\frac{1}{n^4}}{\frac{1}{n^4}} \\
&= \lim_{n\to\infty} \frac{1 + \frac{2}{n^2}}{1 + \frac{3}{n^4}} = 1 = c
\end{aligned}$$

The limit of the ratios of the sequences $c = 1$ is finite and not zero, so the sequences are comparable. This means that the two series $\sum a_n$ and $\sum b_n$ do the same thing: they either both converge or they both diverge. But we already know $\sum b_n$ (the harmonic series) diverges. So the given series $\sum_{n=2}^{\infty} \frac{n^3 - 2n}{n^4 + 3}$ also diverges.