Solutions

Goal: If we know that a power series converges to a specific function, we can manipulate the equation to determine the limits of new power series. This is a nifty and fast way to get lots of new power series representations of functions. Today we will manipulate power series in these ways:

- Substitute
- Multiply by x
- Differentiate
- Integrate

1. Write down a power series representation for the function $f(x) = \dfrac{1}{1-x}$ by using the fact that the geometric series $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$. Write your answer in both expanded form and Σ-notation. On what interval does the series converge to the function?

$$f(x) = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

The series converges to $\frac{1}{1-x}$ on the interval $-1 < x < 1$.

2. Using your response for the last problem, substituting $-x$ in the place of x, find the power series representation for $f(x) = \dfrac{1}{1+x}$. Write your answer in both expanded form and Σ-notation. On what interval does the series converge to the function?

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots = \sum_{n=0}^{\infty} (-x)^n$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$$

Interval of convergence: $-1 < -x < 1 \Rightarrow 1 > x > -1$, so $(-1, 1)$

3. Find the power series representation for $f(x) = \dfrac{1}{1+x^2}$. Write your answer in both expanded form and Σ-notation. On what interval does the series converge to the function?

$\left[\text{Hint: } \dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)}\right]$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Interval of convergence: $-1 < -x^2 < 1 \Rightarrow 1 > x^2 > -1 \Rightarrow -1 < x < 1$

4. Find the power series representation for $\frac{x}{1-x}$. (Hint: multiply answer to problem 1 by x.) On what interval does the series converge to the function?

$$\frac{x}{1-x} = x \cdot \frac{1}{1-x} = x(1+x+x^2+\cdots) = x+x^2+x^3+x^4+\cdots = \sum_{n=0}^{\infty} x^{n+1} = \sum_{n=1}^{\infty} x^n$$

Interval of conv. Same as for $\frac{1}{1-x}$, so $(-1,1)$

5. Find the power series representation for $\frac{1}{(1-x)^2}$. On what interval does the series converge to the function? Hint: Take the derivative of both sides of this equation:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n$$

$$\frac{d}{dx}\left((1-x)^{-1}\right) = 0 + 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = \sum_{n=0}^{\infty} nx^{n-1}$$

$$-(1-x)^{-2}(-1) = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = \sum_{n=0}^{\infty} nx^{n-1}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1} \qquad (\text{Note: } 0 \cdot x^{-1} = 0)$$

Interval of conv: Same as for $\frac{1}{1-x}$, so $(-1,1)$

6. Find the power series representation of $\arctan x$. (Hint: start with the power series for $\frac{1}{1+x^2}$ and antidifferentiate. Solve for the constant of integration by substituting $x = 0$.) On what interval does the series converge to the function?

[Hint: $\arctan x = \int \frac{1}{1+x^2}$]

$$\arctan x = \int 1 - x^2 + x^4 - x^6 + x^8 - \cdots \, dx$$

$$= x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 - \cdots + C$$

$$0 = \arctan(0) = 0 - \frac{0^3}{3} + \frac{0^5}{5} - \frac{0^7}{7} \cdots + C \implies C = 0$$

$$\arctan x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

Interval of conv: Same as for $\frac{1}{1+x^2}$, so $(-1,1)$