Solutions

1. Let's start with the **sequence** $a_n = \frac{1}{n}$. Does this **sequence** converge or diverge? Explain.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$, so $\{a_n\}$ converges to 0.

2. Now consider this infinite **series** (called the harmonic series):

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

(a) Write this series in summation notation.

$\sum_{n=1}^{\infty} \frac{1}{n}$

(b) Now consider this series numerically. Calculate the following partial sums:

$\sum_{n=1}^{1} \frac{1}{n} = 1$

$\sum_{n=1}^{2} \frac{1}{n} = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$

$\sum_{n=1}^{3} \frac{1}{n} \approx 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} = 1.8\overline{3}$

$\sum_{n=1}^{4} \frac{1}{n} \approx 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{11}{6} + \frac{1}{4} = \frac{25}{12} = 2.08\overline{3}$

$\sum_{n=1}^{10} \frac{1}{n} \approx \frac{7381}{2520} \approx 2.92897$

$\sum_{n=1}^{100} \frac{1}{n} \approx 5.1874$

$\sum_{n=1}^{1000} \frac{1}{n} \approx 7.485$

For these I recommend using Wolfram Alpha or a TI84 or something.

(c) Recall that convergence of an infinite series is determined by taking the limit of the partial sums. In other words,

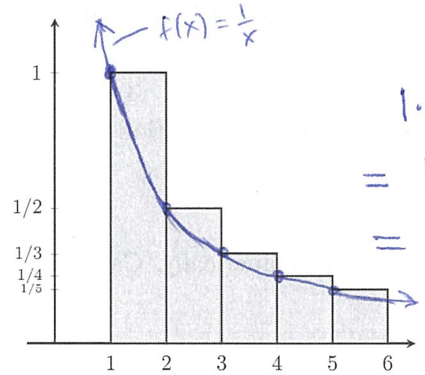
$$\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{n}$$

If the limit is finite, we say the infinite series converges, otherwise we say it diverges. Using what you calculated in part(b) of this problem, make a conjecture about whether or not the harmonic series converges.

The harmonic series appears to diverge because the partial sums look like they are going to ∞. [i.e. $\lim_{N\to\infty} S_N$ dne]

3. The next part of the project introduces the concept of the Integral Test to show a series diverges.

   (a) Every series can be depicted graphically. Write down a sum that gives the area of the shaded region below. How does this sum relate to the harmonic series?



$$1\cdot 1 + 1\cdot \tfrac{1}{2} + 1\cdot\tfrac{1}{3} + 1\cdot\tfrac{1}{4} + 1\cdot\tfrac{1}{5}$$
$$= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5}$$
$$= \sum_{n=1}^{5} \frac{1}{n}.$$

This is the 5th partial sum of the ~~series~~ harmonic series.

   (b) The advantage of representing a series this way is that it can be compared to an improper integral. On the above graph, *carefully* draw the function $f(x) = \frac{1}{x}$.

   (c) How does the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ compare to the improper integral $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$?

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \int_1^{\infty} \frac{1}{x}\,dx$$

Area under $\frac{1}{x}$ is smaller than Area of rectangles.

   (d) Does the improper integral $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ converge or diverge? Calculate it, as a review of improper integrals.

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{T\to\infty} \int_1^{T} \frac{1}{x}\,dx$$
$$= \lim_{T\to\infty} \ln|x|\Big|_1^T$$
$$= \lim_{T\to\infty} (\ln|T| - \ln|1|)$$
$$= \infty$$

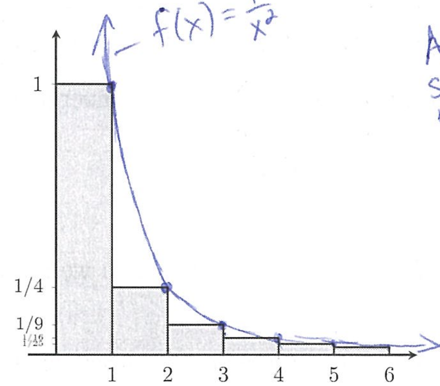
The improper integral $\int_1^{\infty} \frac{1}{x}\,dx$ diverges.

   (e) What can you conclude about the convergence of the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$?

The harmonic series is larger than $\int_1^{\infty} \frac{1}{x}\,dx = \infty$, so it must also diverge to ∞.

4. In the previous problem we compared an infinite series to an improper integral to show divergence of the infinite series. By shifting to the left where we draw the rectangles, we can compare an infinite series to an improper integral to show convergence of the series.

   (a) Write down a sum that gives the area of the shaded region below. How does this sum relate to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$?



Area of shaded Region $= \sum_{n=1}^{6} \frac{1}{n^2}$

This is the 6^{th} partial sum of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

   (b) On the above graph, *carefully* draw the function $f(x) = \frac{1}{x^2}$.

   (c) How does the series $\sum_{n=2}^{\infty} \frac{1}{n^2}$ compare to the improper integral $\int_1^{\infty} \frac{1}{x^2}\, dx$?

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2}\, dx$$

(Area under $\frac{1}{x^2}$ from $x=1$ to ∞ is larger than the area of the rectangles between $x=1$ & ∞)

   (d) Does the improper integral $\int_1^{\infty} \frac{1}{x^2}\, dx$ converge or diverge? Calculate it, as a review of improper integrals.

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{T \to \infty} \int_1^{T} \frac{1}{x^2}\, dx$$
$$= \lim_{T \to \infty} \left. \frac{-1}{x} \right|_1^T$$
$$= \lim_{T \to \infty} \left(\frac{-1}{T} + 1 \right) = 1$$

The integral $\int_1^{\infty} \frac{1}{x^2}\, dx$ converges to 1

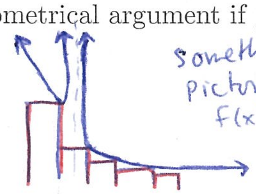
   (e) What can you conclude about the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n^2}$, and thus about the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$?

The series $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, so $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \underbrace{\sum_{n=2}^{\infty} \frac{1}{n^2}}_{\text{finite}}$ must also converge.

5. In problems 3 and 4 we compared infinite series to improper integrals in order to make conclusions about the convergence of divergence of the infinite series. Here is the general result:

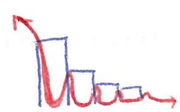
> The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\displaystyle\int_1^{\infty} f(x)\,dx$ converges. In other words:
>
> - if we know $\displaystyle\int_1^{\infty} f(x)\,dx$ *converges*, then we know $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.
> - if we know $\displaystyle\int_1^{\infty} f(x)\,dx$ *diverges*, then we know $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

6. Why do you think we need $f(x)$ to be decreasing? Think about what might go wrong with the geometrical argument if $f(x)$ isn't decreasing.



something like this picture! here Area under f(x) might be infinite (it leaks out) and the Area of the rectangles is maybe finite.

OR



Here, the integral might converge, but the series could diverge. (Rectangle areas much bigger).

7. Now we'll apply the Integral Test in an example. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges or diverges by following these steps:

$a_n = \dfrac{1}{n^2+1}$; so let $f(x) = \dfrac{1}{x^2+1}$

$f(x)$ is decreasing because: (Hint: find $f'(x)$)

$$f'(x) = \frac{d}{dx}\left((x^2+1)^{-1}\right) = -(x^2+1)^{-2}\frac{d}{dx}(x^2) = -(x^2+1)^{-2}\cdot 2x = \frac{-2x}{(x^2+1)^2}$$

$f'(x) = \dfrac{-2x}{(x^2+1)^2} < 0$ for $x \geq 1$, so $f(x)$ is decreasing on $[1,\infty)$.

Check that $f(x)$ is positive:

This is clear:

$f(x) = \dfrac{1}{x^2+1} > 0$ for all x in $[1, \infty)$

We've fulfilled the hypotheses of the integral test, so we can conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ and $\int_1^{\infty} \frac{1}{x^2+1}\,dx$ either both converge or both diverge.

Integrate to determine whether $\int_1^{\infty} \frac{1}{x^2+1}\,dx$ converges or diverges:

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{T\to\infty} \int_1^{T} \frac{1}{x^2+1}\,dx = \lim_{T\to\infty}\left[\arctan x \Big|_1^T\right]$$

$$= \lim_{T\to\infty}\left[\arctan T - \arctan 1\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2}.$$

$\int_1^{\infty} \frac{1}{x^2+1}\,dx$ <u>Converges</u>, and therefore $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also <u>Converges</u>.

8. Use the integral test to determine whether $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ converges or diverges.

Let $f(x) = \frac{1}{x\ln x}$.

- f is continuous on $[2,\infty)$ ✓
- f is positive on $[2,\infty)$ because $x \geq 2 > 0$ and $\ln(x) \geq \ln(2) > 0$ on $[2,\infty)$ ✓
- f is decreasing because x and $\ln(x)$ are both increasing on $[2,\infty)$, so $x\ln(x)$ is increasing, and $\frac{1}{x\ln(x)}$ must be decreasing [can also check that f' is <0]

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx$$

$$= \lim_{T\to\infty} \int_2^{T} \frac{1}{x\ln(x)}\,dx$$

$$= \lim_{T\to\infty} \int_{\ln(2)}^{\ln(T)} \frac{1}{u}\,du$$

$u = \ln(x)$
$du = \frac{1}{x}\,dx$

$$= \lim_{T\to\infty}\left[\ln|u| \Big|_{\ln(2)}^{\ln(T)}\right]$$

$$= \lim_{T\to\infty}\left[\ln|\ln(T)| - \ln|\ln(2)|\right]$$

$$= \infty.$$

Since $\int_2^{\infty} \frac{1}{x\ln x}\,dx$ diverges, the <u>integral test</u> tells us that $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ also <u>diverges</u>