The goal of this worksheet is to recap the basics of differential equations, and to help you see the connections that exist between the graphical, numerical, and analytic ways of viewing them.

1. Using the slope field given below for the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x}$:

   (a) Draw the solution curve that passes through the point $(1, 0)$. Denote this curve by $f(x)$.

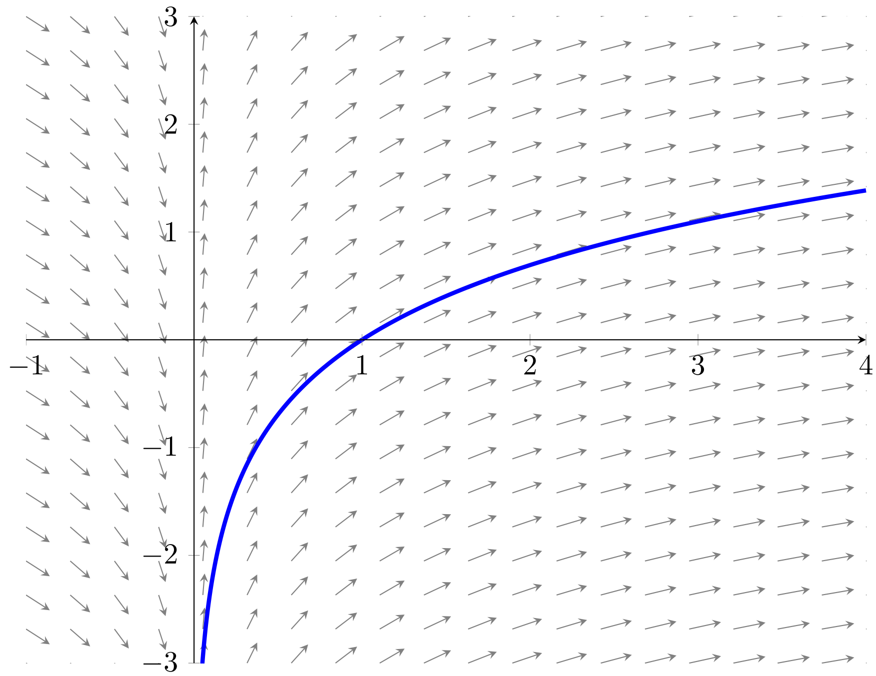
   Solution: See below

   (b) Based on the curve from part (a), make an educated guess as to $\lim_{x\to+\infty} f(x)$.

   Solution: It kind of looks like $\lim_{x\to+\infty} f(x) = +\infty$, although this is not completely clear. It would be reasonable to deduce from the graph that $\lim_{x\to+\infty} f(x) = 2$, say.

   (c) Based on the curve from part (a), make an educated guess as to $\lim_{x\to 0^+} f(x)$.

   Solution: It looks like $\lim_{x\to 0^+} f(x) = -\infty$.



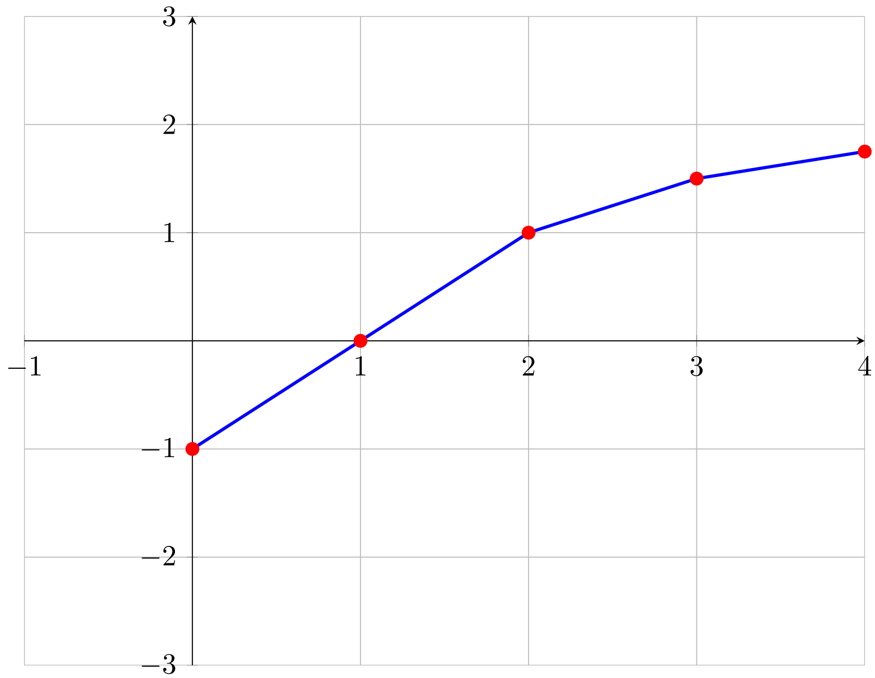
Now we investigate a numerical way of constructing solution curves, called Euler's Method.

2. Given the differential equation $dy/dx = 1/x$ and the coordinate plane below, do the following:

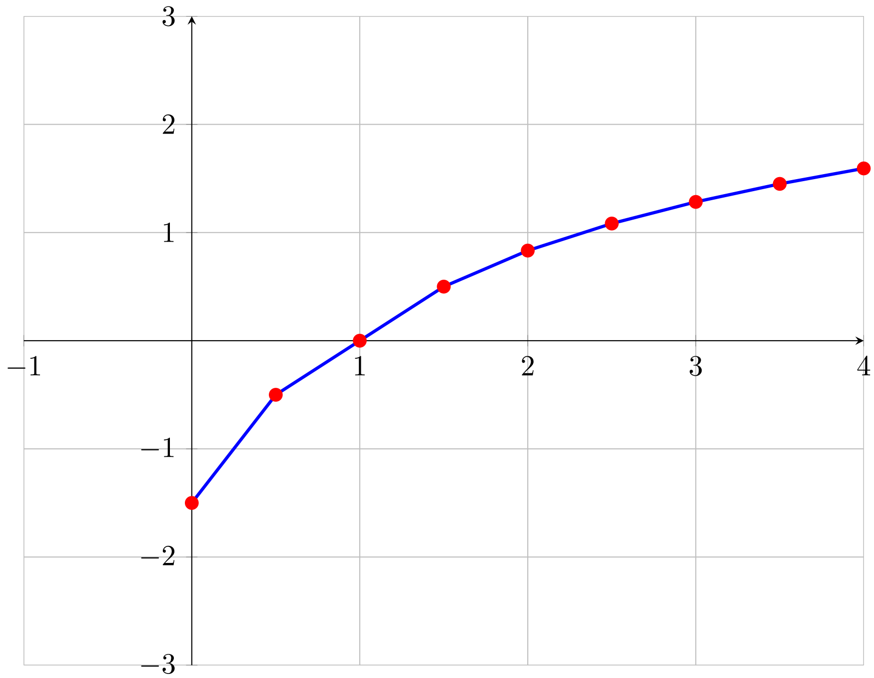
   (a) Start at the point $(1, 0)$. Call this point (x_0, y_0), and mark it with a dot. Using the given differential equation, evaluate dy/dx at this point. This value of dy/dx is the slope of the tangent line to the solution curve at the point $(1, 0)$. If you travel along this tangent line for a *horizontal* distance of one unit, at what point in the xy-plane do you end up? Call this new point $(x_1 = 2, y_1)$ and mark it with a dot.

   (b) Start at the point $(x_1 = 2, y_1)$ that you found above. Evaluate dy/dx at this point. Repeat the process of part (a) (that is, start at (x_1, y_1) and move along your tangent line for a horizontal distance of one unit) to get a new point $(x_2 = 3, y_2)$. Mark it with a dot. Do this one more time in this direction to get a fourth point $(x_3 = 4, y_3)$.

   (c) Finally, start at the original point $(x_0, y_0) = (1, 0)$. This time, *decrease* the value of x by 1, and move along the tangent line, to find the point (x_{-1}, y_{-1}). Mark this point. You should have five dots in your coordinate plane. Connect the dots with straight lines.



3. Now repeat the process of problem 2, but this time taking step sizes in the x-direction of length 0.5 instead of length 1. In other words, start at the point $(1, 0)$, and then find $(x_1 = 1.5, y_1), (2, y_2), (2.5, y_3), (3, y_4), (3.5, y_5), (4, y_6)$. Then start again at $(1, 0)$, but move backwards to get $(.5, y_{-1})$ and $(0, y_{-2})$. Mark your points with dots on the coordinate plane below, and connect the dots with straight lines.



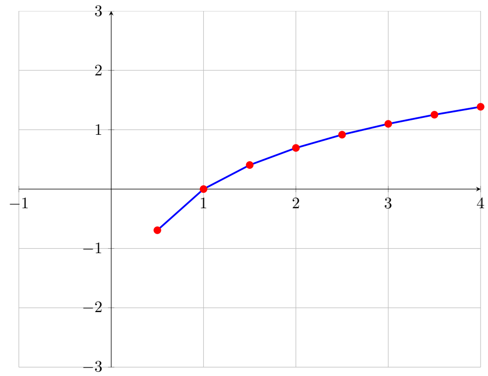
4. (a) Compare the curve you drew for problem 1 with the polygonal paths you drew for problems 2 and 3. What do these three curves have in common? What about them is different?

   Solution: All three are similar. The curve in problems 2 and 3 have a finite value at $x = 0$; the curve in problem 1 does not. And of course the curve in problem 1 is smoother. The polygonal path in problem 3 looks more like that of problem 1 than does the curve of problem 2.

   (b) Based on your answers above, which of the two polygonal curves is the better approximation to the solution curve you sketched in problem 1? Why do you think that is?

   Solution: The curve in problem 3. This is probably because we chose a smaller "step size." This means we'll have less error in our approximation at each step .

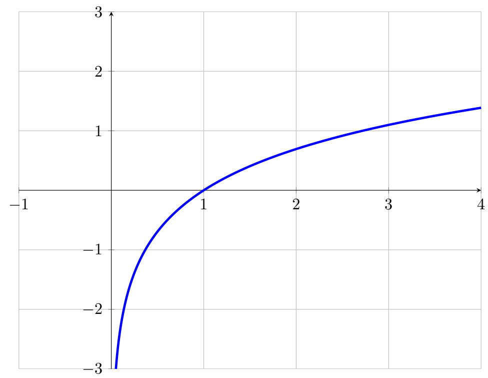
5. (a) Using your calculator, calculate $y = \ln x$ for $x = 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4$. Plot these points on the coordinate plane below, and connect the dots. Does this look familiar?



Solution: It looks like the previous plots found using Euler's method for $\dfrac{dy}{dx} = \dfrac{1}{x}$

(b) Confirm that $f(x) = \ln x$ is a solution to the initial value problem $dy/dx = 1/x$ with initial condition $f(1) = 0$. Plot the solution curve satisfying $x > 0$ on the coordinate plane below. Does this look familiar?

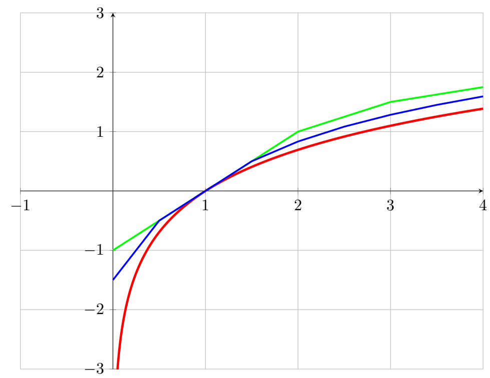
Solution: $f'(x) = 1/x$, so yes it satisfies $\dfrac{dy}{dx} = \dfrac{1}{x}$. $f(1) = \ln 1 = 0$, so yes it satisfies the initial condition.



Solution: It looks like the previous plot where we connected values of $\ln(x)$.

6. Given everything that you've done above, how would you use limits to pass from the polygonal paths you constructed in problems 2 and 3 to the solution curve you found in problem 5?

   Solution: What we've seen above suggests that, if we use a smaller and smaller "step size," then our approximate solution (constructed by moving along tangent lines) to the given differential equation plus initial condition gets closer and closer to the actual solution. Hopefully, in the limit as step size$\to 0$, the approximate solution should BECOME the actual one.
   For the sake of comparison, here is the function $y = \ln x$ (in RED), together with the polygonal approximations found in problems 2 (GREEN) and 3 (BLUE). Cool!



7. Now consider the more general situation, estimating a solution to $\frac{dy}{dx} = F(x, y)$, starting at an initial point (x_0, y_0), using a step-size of h. Write a recursive equation that shows how to get y_{n+1} from x_n and y_n, and how to get x_{n+1} from x_n. This method for numerically estimating a solution to a differential equation is called Euler's method.

   Solution: In problem 3, we found y_1 from $y_0 = 0$ by doing the following: $y_1 = 0 + .5 \cdot \dfrac{1}{1}$, where .5 is a set horizontal step-size, and $\dfrac{1}{1}$ is the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x}$ evaluated at $x_0 = 1$. This suggests the recursive process:

$$y_{n+1} = y_n + h \cdot F(x_n, y_n) \quad \text{where } h \text{ is the horizontal step-size.}$$

   x_n starts at x_0 and then moves along the horizontal axis one step-size for each integer value of n, giving us:

$$x_{n+1} = x_0 + h \cdot (n + 1)$$

8. If you happen to have a TI-83 or TI-84, there is a slick way to use a graphing calculator to implement the recursive equations from the last problem. As an example, re-do problem 3 this way:
   1 STO X (store 1 in the variable x)
   0 STO Y (store 0 in the variable y)
   Y+.5(1/X) STO Y : (X+.5) STO X : {X,Y}
   (calculate the new value of y and store it, calculate the new value of x and store it, output both of these new values). Here's the slick part: the last step can be repeated indefinitely (you can do this without re-entering by hitting 2nd ENTRY). Try it, making sure it matches the results of problem 3.

   Solution: The calculator gives me:

   $$(1.5, 0.5), (2, 0.8\overline{33}), (2.5, 1.08\overline{33}), (3, 1.28\overline{33}), (3.5, 1.45), (4, 1.5929...)$$

   This matches what I graphed in problem 3.

9. Use your calculator to perform Euler's method with a step size of 0.1 to estimate the value of y when $x = 1$ for the initial value problem $y' = x + y$, $y(0) = 1$.

   Solution: This time the calculator gives me:

   $$(0.1, 1.1), (0.2, 1.22), (0.3, 1.362), (0.4, 1.5282), (0.5, 1.72102), ...$$

   $$...(0.6, 1.943122), (0.7, 2.1974342), (0.8, 2.48717762), (0.9, 2.815895382), (1, 3.18748492)$$

   So my estimate is $y(1) \approx 3.18748492$.

10. Go to the website `http://www.math.rutgers.edu/~sontag/JODE/JOdeApplet.html`
    Enter the differential equation from the last problem, and use the tool to check whether your answer to problem 9 is an underestimate or overestimate.

    Solution: It is an underestimate.

11. Use the slope field tool from the last problem to graph the slope field for the logistic differential equation $\dfrac{dP}{dt} = .1P(10 - P)$. Then graph the solutions for the initial values given by the points $(0, 0)$, $(1, 4)$, $(4, 1)$, $(-5, 1)$, $(-2, 12)$, and $(-2, 10)$.