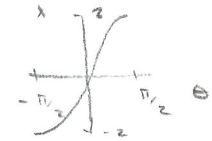
§ 5.7: Integration via Trig Sub.

It is best to use trig sub to integrate expressions that involve $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, or $\sqrt{a^2+x^2}$ for 'a' a constant.

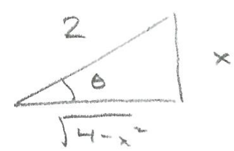
Example 1: Integrate

$$\int \frac{1}{\sqrt{4-x^2}}\,dx$$

Domain: $(-2, 2)$



First, note the right triangle



and observe that $x = 2\sin\theta$ with $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Now $\frac{dx}{d\theta} = 2\cos\theta$ so $dx = 2\cos\theta\, d\theta$

Then,

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{4-(2\sin\theta)^2}}\,2\cos\theta\, d\theta$$

$$= \int \frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}}\,d\theta = \int \frac{2\cos\theta}{\sqrt{4(1-\sin^2\theta)}}\,d\theta$$

$$= \int \frac{2\cos\theta}{2\sqrt{\cos^2\theta}}\,d\theta = \int \frac{2\cos\theta}{2\cos\theta}\,d\theta = \int 1\, d\theta$$

$= \theta + C$

Now $x = 2\sin\theta$, so $\frac{x}{2} = \sin\theta \implies \arcsin\left(\frac{x}{2}\right) = \theta$

since $\theta \in [-\pi/2, \pi/2]$.

So $\theta + C = \arcsin\left(\frac{x}{2}\right) + C$

In general, to simplify $\sqrt{a^2 - x^2}$ for constant 'a'

use the substitution $x = a\sin\theta$ with $\theta \in [-\pi/2, \pi/2]$.

Example 2: Sometimes you have to complete the square

before you make the substitution. Find

$$\int \frac{4}{\sqrt{2z - z^2}}\, dz$$

Now $-z^2 + 2z = -(z^2 - 2z) = -(z^2 - 2z + 1 - 1)$

$= -(z-1)^2 + 1 = 1 - (z-1)^2$

Hence,
$$\int \frac{4}{\sqrt{2z - z^2}}\, dz = \int \frac{4}{\sqrt{1^2 - (z-1)^2}}\, dz \qquad (*)$$

Let $z - 1 = \sin\theta$. Then $z = 1 + \sin\theta$.

$$\frac{dz}{d\theta} = \cos\theta$$

Now

$$(*) = \int \frac{4}{\sqrt{1 - \sin^2\theta}} \cos\theta \, d\theta = \int \frac{4\cos\theta}{\cos\theta} \, d\theta$$

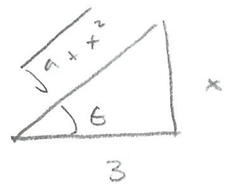
$$= 4\theta + C$$

But $\arcsin(z-1) = \theta$ so

$$4\theta + C = 4\arcsin(z-1) + C.$$

Example 3: Find $\int \frac{1}{9+x^2} \, dx$ using an appropriate trig sub.

Observe that



Let $x = 3\tan\theta$. Then $\frac{dx}{d\theta} = \frac{3}{\cos^2\theta}$

Hence, $dx = \frac{3\,d\theta}{\cos^2\theta}$. Then

$$\int \frac{1}{9+x^2}\,dx = \int \frac{1}{9+(3\tan\theta)^2}\,\frac{3}{\cos^2\theta}\,d\theta$$

$$= \int \frac{3}{9(1+\tan^2\theta)\cos^2\theta}\,d\theta = \int \frac{3}{9\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right)\cos^2\theta}\,d\theta$$

$$= \int \frac{3}{9(\cos^2\theta+\sin^2\theta)}\,d\theta = \int \frac{3}{9}\,d\theta = \frac{1}{3}\theta + C$$

Since $\frac{x}{3} = \tan\theta$ for $-\pi/2 < \theta < \pi/2$,

$$\arctan\left(\frac{x}{3}\right) = \theta$$

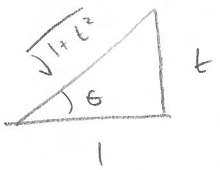
So $\frac{1}{3}\theta + C = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C.$

Using this example as motivation, to simplify a^2+x^2 or $\sqrt{a^2+x^2}$ for constant a, use the substitution

$x = a\tan\theta$ with $-\pi/2 < \theta < \pi/2$.

Example 4: Use a tangent substitution to find

$$\int \frac{1}{t^2\sqrt{1+t^2}}\,dt$$



$t = \tan\theta$ for $-\pi/2 < \theta < \pi/2$.

$$\frac{dt}{d\theta} = \frac{1}{\cos^2\theta}$$

$$dt = \frac{1}{\cos^2\theta}\,d\theta$$

So $$\int \frac{1}{t^2\sqrt{1+t^2}}\,dt = \int \frac{1}{\tan^2\theta\sqrt{1+\tan^2\theta}}\cdot\frac{1}{\cos^2\theta}\,d\theta$$

$$= \int \frac{1}{\frac{\sin^2\theta}{\cancel{\cos^2\theta}}\sqrt{1+\frac{\sin^2\theta}{\cos^2\theta}}}\cdot\frac{1}{\frac{\cancel{\cos^2\theta}}{1}}\,d\theta$$

$$= \int \frac{1}{\sin^2\theta\sqrt{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}}\,d\theta$$

$$= \int \frac{1}{\sin^2\theta\sqrt{\frac{1}{\cos^2\theta}}}\,d\theta$$

$$= \int \frac{1}{\sin^2\theta \cdot \frac{1}{\cos\theta}} \qquad \left(\text{since } \cos\theta > 0 \text{ on } -\pi/2 < \theta < \pi/2 \right)$$

$$= \int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$$= -\frac{1}{\sin\theta} + C$$

Now $\theta = \arctan t$ so

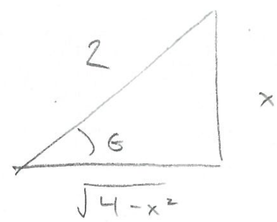
$$\frac{1}{\sin\theta} + C = -\frac{1}{\sin(\arctan t)} + C$$

$$= -\frac{1}{\frac{t}{\sqrt{1+t^2}}} + C \qquad \left(\text{going back to right triangle} \right)$$

$$= -\frac{\sqrt{1+t^2}}{t} + C.$$

Ex: Use a sine sub to find

$$\int \sqrt{4-x^2}\,dx$$



$x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4-(2\sin\theta)^2}\; 2\cos\theta\, d\theta$$

$$= \int \sqrt{4-4\sin^2\theta} \cdot 2\cos\theta\, d\theta$$

$$= \int 2\sqrt{1-\sin^2\theta}\;\, 2\cos\theta\, d\theta$$

$$= \int 2\sqrt{\cos^2\theta} \cdot 2\cos\theta\, d\theta$$

$$= \int 4\cos^2\theta\, d\theta$$

$$= 4\left(\frac{\theta + \cos\theta\sin\theta}{2}\right) + 2$$

$$= 2\left(\arcsin\left(\frac{x}{2}\right) + \frac{\sqrt{4-x^2}}{2}\cdot\frac{x}{2}\right) + 2$$

$$= 2\arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + 2$$