§ 5.7: Partial Fractions

A polynomial $p(x)$ is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

The degree of the polynomial, denoted $\deg p(x)$, is n.

A rational function is a function of the form

$$r(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

Partial fractions is a technique used to integrate rational functions, where $\deg p(x) < \deg q(x)$

Example 1: Integrate

$$\int \frac{1}{(x-2)(x-5)}\,dx$$

Note: $p(x) = 1$ and $q(x) = (x-2)(x-5)$

Write

$$\frac{1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5} \qquad (1)$$

We will now determine the constants A and B.

Multiply both sides of (1) by $(x-2)(x-5)$ to get

$$1 = A(x-5) + B(x-2)$$

$$1 = (A+B)x - 5A - 2B$$

$$0x + 1 = (A+B)x + (-5A-2B)$$

Hence,
$$0 = A + B$$
$$1 = -5A - 2B$$

Solving these equations:

$$A + B = 0 \implies -A = B$$

$$-5A - 2B = 1 \implies -5A - 2(-A) = 1$$

$$\implies -3A = 1 \implies A = -\frac{1}{3}$$

Then $B = \frac{1}{3}$. Hence,

$$\frac{1}{(x-2)(x-5)} = \frac{-1/3}{x-2} + \frac{1/3}{x-5} = -\frac{1}{3}\left(\frac{1}{x-2}\right) + \frac{1}{3}\left(\frac{1}{x-5}\right)$$

So
$$\int \frac{1}{(x-2)(x-5)}\,dx = -\frac{1}{3}\int \frac{1}{x-2}\,dx + \frac{1}{3}\int \frac{1}{x-5}\,dx$$

$$= -\frac{1}{3}\ln|x-2| + \frac{1}{3}\ln|x-5| + C$$

In general, a similar decomposition structure works whenever

① The denominator factors into <u>distinct</u> linear roots

② $\deg p(x) < \deg q(x)$

<u>Example</u> 2 : Integrate

$$\int \frac{x}{(x-1)^2}\,dx$$

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{x-1} \quad \text{(doesn't work)}$$

$$x = A(x-1) + B(x-1)$$

$$x = (A+B)x + (-A-B)$$

$$A+B = 1 \qquad -A-B = 0$$

$$-(A+B) = 0 \qquad \text{no soln!}$$

Note: $p(x) = x$, $q(x) = (x-1)^2$ (it has a repeated factor)

In this case, use the following decomposition structure

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \qquad (2)$$

Multiply both sides of (2) by $(x-1)^2$ to get

$$x = A(x-1) + B$$

$$x = Ax - A + B$$

$$1x + 0 = Ax + (-A+B)$$

$$A = 1 \qquad 0 = -A + B$$

$$A = B$$

So $A = 1$ and $B = 1$

Hence,

$$\int \frac{x}{(x-1)^2}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$$= \ln|x-1| - \frac{1}{x-1} + C$$

In general, a similar decomposition structure works whenever

① The denominator factors into repeated linear factors

② $\deg p(x) < \deg q(x)$

<u>Example</u> 3: Integrate

$$\int \frac{x+1}{(x^2+1)(x-1)}\,dx$$

<u>Note</u>: $p(x) = x+1$ and $q(x) = x^2+1$ ($q(x)$ cannot be factored into linear factors)

In this case, use the decomposition structure:

$$\frac{x+1}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} \qquad (3)$$

Multiply both sides of (3) by $(x^2+1)(x-1)$ to get:

$$x+1 = (Ax+B)(x-1) + C(x^2+1)$$

$$x+1 = (A+C)x^2 + (B-A)x + (C-B)$$

$A + C = 0$ $\qquad$ $B - A = 1$ $\qquad$ $C - B = 1$

$A = -C$ $\qquad$ $B + C = 1$ $\qquad$ $C - (1-C) = 1$

$B = 1 - C$ $\qquad$ $2C - 1 = 1$

$C = 1$

So $A = -1$, $B = 0$, and $C = 1$.

Hence,
$$\int \frac{x+1}{(x^2+1)(x-1)}\,dx = \int \frac{-x+0}{x^2+1}\,dx + \int \frac{1}{x-1}\,dx$$

$$= -\frac{1}{2}\ln|x^2+1| + \ln|x-1| + C$$

What to do if $\deg p(x) \geq \deg q(x)$? Use polynomial long division

Example 4: Compute

$$\int \frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10}\,dx$$

Use polynomial long division:

$$\begin{array}{r|l} & x \\ \hline x^2 - 7x + 10 & x^3 - 7x^2 + 10x + 1 \\ & -\ x^3 - 7x^2 + 10x \\ \hline & \qquad\qquad\qquad\qquad 1 \end{array}$$

Hence,

$$\frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10} = x + \underbrace{\frac{1}{x^2 - 7x + 10}}_{(*)}$$

One can then use partial fractions to decompose $(*)$.