The goal of this project is to develop "function sense" about the decay rate of functions. This skill is important for determining convergence of improper integrals, and it will become important again when we study convergence of series.

Problems 1-4 will help develop your *numerical* "function sense."

1. Consider the improper integral $\displaystyle\int_3^\infty \frac{1}{x^2\ln(x)}dx$. Use technology to find:

   (a) $\displaystyle\int_3^{10} \frac{1}{x^2\ln(x)}dx =$

   (b) $\displaystyle\int_3^{100} \frac{1}{x^2\ln(x)}dx=$

   (c) $\displaystyle\int_3^{1000} \frac{1}{x^2\ln(x)}dx=$

2. What can you conclude about the convergence/divergence of the improper integral $\int_3^\infty \frac{1}{x^2\ln(x)}\,dx$ based on your data from problem 1?

   (a) Based on numerical evidence, the above integral converges.
   (b) Based on numerical evidence, the above integral diverges.
   (c) Based on numerical evidence, it appears that the above integral converges.
   (d) Based on numerical evidence, it appears that the above integral diverges.

3. Now, consider the improper integral $\displaystyle\int_3^\infty \frac{\ln(x)}{\sqrt{x}}dx$. Again, use technology to find

   (a) $\displaystyle\int_3^{10} \frac{\ln(x)}{\sqrt{x}}dx =$

   (b) $\displaystyle\int_3^{100} \frac{\ln(x)}{\sqrt{x}}dx =$

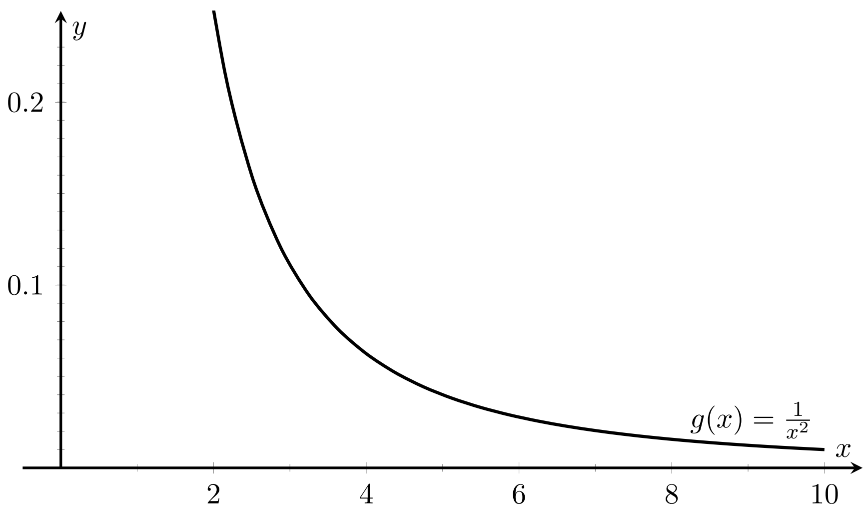
   (c) $\displaystyle\int_3^{1000} \frac{\ln(x)}{\sqrt{x}}dx =$

4. What can you conclude about the convergence/divergence of the improper integral $\int_3^\infty \frac{\ln(x)}{\sqrt{x}}\,dx$ based on your data from problem 3?

   (a) Based on numerical evidence, the above integral converges.
   (b) Based on numerical evidence, the above integral diverges.
   (c) Based on numerical evidence, it appears that the above integral converges.
   (d) Based on numerical evidence, it appears that the above integral diverges.

Problems 5-8 will help develop your *graphical* "function sense."

5. Sketch the graph of $f(x) = \dfrac{1}{x^2 \ln(x)}$. The graph of $g(x) = \dfrac{1}{x^2}$ has been provided for reference. In your sketch, identify any intersection points and pay attention to which function has higher values.

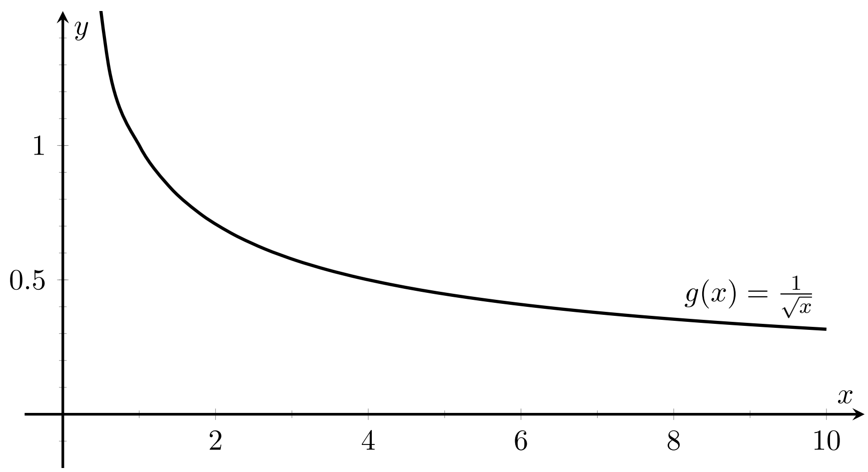


6. Using the graph you just made:

   (a) Do you think $\displaystyle\int_3^\infty \frac{1}{x^2 \ln(x)}\,dx \quad \leq \quad \int_3^\infty \frac{1}{x^2}\,dx$ is true? Why or why not?

   (b) What does this suggest about the convergence/divergence of the two integrals and why?

7. Sketch the graph of $f(x) = \dfrac{\ln(x)}{\sqrt{x}}$. The graph of $g(x) = \dfrac{1}{\sqrt{x}}$ has been provided for reference. In your sketch, identify any intersection points, and pay attention to which function is higher. Do these functions have aysmptotes?



8. Using the graph you just made

   (a) Do you think $\displaystyle\int_3^\infty \frac{\ln(x)}{\sqrt{x}}\,dx \quad \geq \quad \int_3^\infty \frac{1}{\sqrt{x}}\,dx$ is true? Why or why not?

   (b) What does this suggest about the convergence/divergence of the two integrals and why?

9. What advice would you give to a classmate who says "in Problem 8, we found that the function $f(x) = \frac{\ln(x)}{\sqrt{x}}$ diverges"?

10. The punchline:

Comparison Theorem for Integrals
If f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$, then

(a) If ________________ is convergent, then ________________ is convergent.

(b) If ________________ is divergent, then ________________ is divergent.

If we know $\int_a^\infty f(x)dx$ diverges, what can we conclude about $\int_a^\infty g(x)dx$?

And if we know $\int_a^\infty g(x)dx$ converges, what can we conclude about $\int_a^\infty f(x)dx$?

Putting it into practice:

11. Consider the integral $\displaystyle\int_2^\infty \frac{\cos^2(x)}{x^2}dx$.

 (a) Do you think this improper integral converges or diverges?

 (b) A good comparison function is:

 (c) Write down the inequality that compares $\dfrac{\cos^2(x)}{x^2}$ to your answer to (b).

 (d) How will this inequality help you prove that your guess is correct?

12. Consider the integral $\displaystyle\int_1^\infty \frac{1+\sin^4(2x)}{\sqrt{x}}dx$.

 (a) Do you think this improper integral converges or diverges?

(b) A good comparison function is:

(c) Write down the inequality that compares $\dfrac{1+\sin^4(2x)}{\sqrt{x}}$ to your answer to (b).

(d) How will this inequality help you prove that your guess is correct?

13. Consider the integral $\displaystyle\int_2^\infty \frac{1}{x+e^x}dx$.

(a) Do you think this improper integral converges or diverges?

(b) A good comparison function is:

(c) Write down the inequality that compares $\dfrac{1}{x+e^x}$ to your answer to (b).

(d) How will this inequality help you prove that your guess is correct?

14. Consider the integral $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2-1}}dx$.

(a) Do you think this improper integral converges or diverges?

(b) A good comparison function is:

(c) Write down the inequality that compares $\dfrac{1}{\sqrt{x^2-1}}$ to your answer to (b).

(d) How will this inequality help you prove that your guess is correct?

The limits and limitations of your "function sense."

15. Consider the integral $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2+1}}dx$

    (a) Do you think this improper integral converges or diverges?

    (b) A good comparison function is:

    (c) Write down an inequality that compares $\dfrac{1}{\sqrt{x^2+1}}$ to your answer to (b).

    (d) How will this inequality help you prove your guess? (or will it?)

16. If your attempt at comparison above failed, then make a guess about the convergence/divergence of this integral by investigating the above integral numerically.

17. Even though $\dfrac{1}{x}$ did not produce the desired inequality, it may still provide a useful comparison.

    (a) Show that $\displaystyle\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x^2+1}}}{\frac{1}{x}} = 1$

    (b) What does this say about the relationship between $\frac{1}{\sqrt{x^2+1}}$ and $\dfrac{1}{x}$?

18. Try to write your own theorem that involves the limit idea above. I will get you started.

The Limit Comparison Test: If $f(x)$ and $g(x)$ are positive continuous functions, and $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} =$ __________, then: