FINAL EXAM
CALCULUS 2

MATH 2300
FALL 2018

Name

PRACTICE EXAM

SOLUTIONS

Please answer all of the questions, and show your work.
You must explain your answers to get credit.
You will be graded on the clarity of your exposition!

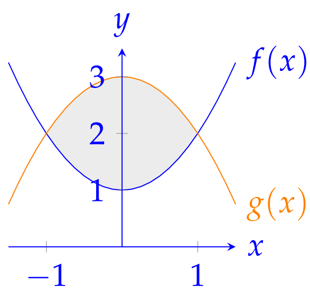
Date: November 26, 2018.

1
10 points

1. Consider the region bounded by the graphs of $f(x) = x^2 + 1$ and $g(x) = 3 - x^2$.

1.(a). *(5 points)* Write the integral for the volume of the solid of revolution obtained by rotating this region about the x-axis. Do not evaluate the integral.

SOLUTION: We can see the region in question below.



Using the washer method, the volume integral is

$$\pi \int_{-1}^{1} g(x)^2 - f(x)^2 \, dx = \pi \int_{-1}^{1} (3 - x^2)^2 - (x^2 + 1)^2 \, dx.$$

1.(b). *(5 points)* Write the integral for the volume of the solid of revolution obtained by rotating this region about the line $x = 3$. Do not evaluate the integral.

SOLUTION: Again using the washer method, the integral is equal to

$$\pi \int_{-1}^{1} (3 - f(x))^2 - (3 - g(x))^2 \, dx = \pi \int_{-1}^{1} (2 - x^2)^2 - x^4 \, dx.$$

2
6 points

2. MULTIPLE CHOICE: Circle the best answer.

2.(a). *(1 point)* Is the following integral an improper integral?

Yes No

2.(b). *(5 points)* Evaluate the integral. $\int_{-1}^{1} \frac{1}{x^2}\, dx =$

SOLUTION: The function $1/x^2$ is undefined at $x = 0$, so we we must evaluate the improper integral as a limit.

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{x^2}\, dx &= \lim_{c\to 0^-} \int_{-1}^{c} \frac{1}{x^2}\, dx + \lim_{c\to 0^+} \int_{c}^{1} -\frac{1}{x^2}\, dx \\
&= \lim_{c\to 0^-} -\frac{1}{x}\bigg|_{-1}^{c} + \lim_{c\to 0^+} -\frac{1}{x}\bigg|_{c}^{1} \\
&= \lim_{c\to 0^-} -\left(\frac{1}{c} - \frac{1}{-1}\right) + \lim_{c\to 0^+} -\left(\frac{1}{1} - \frac{1}{c}\right) \\
&= \lim_{c\to 0^-} -\left(\frac{1}{c} + 1\right) + \lim_{c\to 0^+} -\left(1 - \frac{1}{c}\right).
\end{aligned}$$

Now, since

$$\lim_{c\to 0^-} -\left(\frac{1}{c} + 1\right) = \lim_{c\to 0^-} \frac{-1}{c} - 1$$

and

$$\lim_{c\to 0^+} -\left(1 - \frac{1}{c}\right) = \lim_{c\to 0^+} \frac{1}{c} - 1$$

both diverge to ∞, and so the integral does not converge. Thus, the integral diverges.

3
14 points

3. Consider the curve parameterized by $\begin{cases} x = \frac{1}{3}t^3 + 3t^2 + \frac{2}{3} \\ y = t^3 - t^2 \end{cases}$ for $0 \leq t \leq \sqrt{5}$.

3.(a). *(6 points)* Find an equation for the line tangent to the curve when $t = 1$.

SOLUTION: We first find a general formula for the slope using the chain rule, and then evaluate at $t = 1$, giving

$$\left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=1} = \left.\frac{3t^2 - 2t}{t^2 + 6t}\right|_{t=1} = \frac{1}{7}.$$

Since $x(1) = 4$ and $y(1) = 0$, we need the formula for a line with slope $1/7$ that passes through $(4,0)$. This equation is

$$y = \frac{1}{7}x - \frac{4}{7}$$

3.(b). *(3 points)* Compute $\frac{d^2y}{dx^2}$ at $t = 1$.

SOLUTION: Again employing the chain rule,

$$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \left.\frac{d}{dx}\frac{dy}{dx}\right|_{t=1} = \left.\frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}}\right|_{t=1} = \left.\frac{\frac{(6t-2)(t^2+6t)-(2t+6)(3t^2-2t)}{(t^2+6t)^2}}{t^2+6t}\right|_{t=1} = \frac{20}{7^3}.$$

3.(c). *(5 points)* Write an integral to compute the total arc length of the curve. Do not evaluate the integral.

SOLUTION: Arc length is given by

$$\int_0^{\sqrt{5}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\sqrt{5}} \sqrt{(t^2+6t)^2 + (3t^2-2t)^2}\, dt.$$

4
8 points

4. Consider the function $f(x) = x^2 \arctan(x)$.

4.(a). *(5 points)* Find a power series representation for $f(x)$.

SOLUTION: The power series of $\arctan(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, with interval of convergence $x \in [-1, 1]$. Thus,

$$f(x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{2n+1}$$

for $x \in [-1, 1]$.

4.(b). *(3 points)* What is $f^{(83)}(0)$, the 83rd derivative of $f(x)$ at $x = 0$?

SOLUTION: Using the above power series representation,

$$f^{(83)}(x) = \sum_{n=40}^{\infty} \frac{(2n+3)!}{(2n+3-83)!} \frac{(-1)^n x^{2n+3-83}}{2n+1} = \sum_{n=40}^{\infty} \frac{(2n+3)!}{(2(n-40))!} \frac{(-1)^n x^{2(n-40)}}{2n+1}.$$

Thus,

$$f^{(83)}(0) = (83)! \frac{(-1)^{40}}{2 * 40 + 1} = 83 * 82 * (80!).$$

5. A tank contains 200 L of salt water with a concentration of 4 g/L. Salt water with a concentration of 3 g/L is being pumped into the tank at the rate of 8 L/min, and the tank is being emptied at the rate of 8 L/min. Assume the contents of the tank are being mixed thoroughly and continuously. Let $S(t)$ be the amount of salt (measured in grams) in the tank at time t (measured in minutes).

5
10 points

5.(a). *(1 points)* What is the amount of salt in the tank at time $t = 0$?

SOLUTION: $S(0)\text{g} = 200\text{L} \cdot 4\text{g/L} = 800$ g.

5.(b). *(2 points)* What is the rate at which salt enters the tank?

SOLUTION: $8\text{L/min} \cdot 3\text{g/L} = 24$ g/min

5.(c). *(2 points)* What is the rate at which salt leaves the tank at time t?

SOLUTION: As the volume of water is a constant 200 L, this is $\frac{S(t)\text{g}}{200\text{L}} \frac{8\text{L}}{\text{min}} = \frac{S(t)}{25} \frac{\text{g}}{\text{min}}$.

5.(d). *(1 points)* What is $\frac{dS}{dt}$, the net rate of change of salt in the tank at time t?

SOLUTION: Net change is given by gain minus loss, so using parts (b) and (c),

$$\frac{dS}{dt} \frac{\text{g}}{\text{min}} = 24 - \frac{S(t)}{25} \frac{\text{g}}{\text{min}}$$

5.(e). *(4 points)* Write an initial value problem relating $S(t)$ and $\frac{dS}{dt}$. Solve the initial value problem.

SOLUTION: The initial value problem is $\frac{dS}{dt} = 24 - \frac{S(t)}{25}$, with $S(0) = 800$. Since this differential equation is separable, we can solve by separating and then integrating:

$$\int \frac{1}{24 - \frac{1}{25}S}\, dS = \int dt$$

$$-25 \ln \left| 24 - \frac{1}{25}S \right| = t + C,$$

Note that $24 - \frac{1}{25}S \leq 0$, so we can write this as $-25 \ln \left(\frac{1}{25}S - 24 \right) = t + C$, so that $\frac{1}{25}S - 24 = Ae^{-\frac{1}{25}t}$. From this we get $S = Ae^{-\frac{1}{25}t} + 600$. Setting $t = 0$, and using (a), we find the answer is

$$S = 200e^{-\frac{1}{25}t} + 600$$

6
8 points

6. Compute the following integrals.

6.(a). *(4 points)* $\int \sin^3(x)\cos^2(x)\,dx$

SOLUTION: First, using the pythagorean identity,

$$\begin{aligned}\int \sin^3(x)\cos^2(x)\,dx &= \int \sin(x)(1-\cos^2(x))\cos^2(x)\,dx\\ &= \int \sin(x)\cos^2(x)\,dx - \int \sin(x)\cos^4(x)\,dx.\end{aligned}$$

Now, let $u = \cos(x)$, so that $du = -\sin(x)\,dx$. Then the above equation is equal to

$$\int -u^2\,du + \int u^4\,du = -\frac{u^3}{3} + \frac{u^5}{5} + C.$$

Finally, reversing our substitution, we find that

$$\int \sin^3(x)\cos^2(x)\,dx = -\frac{\cos^3(x)}{3} + \frac{\cos^5(x)}{5} + C.$$

6.(b). *(4 points)* $\int \frac{x+1}{x^2(x-1)}\,dx$

SOLUTION: We start by using partial fractions:

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1},$$

which gives

$$x+1 = Ax(x-1) + B(x-1) + Cx^2 = (A+C)x^2 + (B-A)x - B,$$

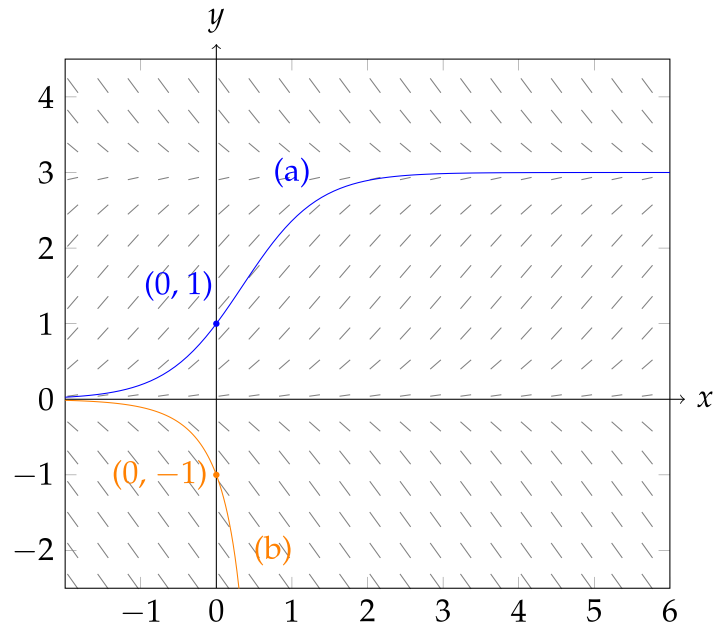
and so $B = -1$, $A = -2$, and $C = 2$. Thus,

$$\begin{aligned}\int \frac{x+1}{x^2(x-1)}\,dx &= \int \frac{-2}{x} + \frac{-1}{x^2} + \frac{2}{x-1}\,dx\\ &= -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C.\end{aligned}$$

7
6 points

7. A slope field for the differential equation $y' = 2y\left(1 - \frac{y}{3}\right)$ is shown below.

$$y' = 2y\left(1 - \frac{y}{3}\right)$$



7.(a). *(2 points)* Sketch the graph of the solution that satisfies following initial condition. Label the solution as (a).

$$y(0) = 1$$

7.(b). *(2 points)* Sketch the graph of the solution that satisfies following initial condition. Label the solution as (b).

$$y(2) = -1$$

7.(c). *(2 points)* Given the initial condition $y(0) = c$, for what values of c is $\lim_{x\to\infty} y(x)$ finite?

SOLUTION: $\lim_{x\to\infty} y(x)$ is finite when $c \geq 0$. This can be seen in the following facts: if $c = 0, 3$, then $y' = 0$ and so $y(x) = c$, while if $c \in (0, 3)$, $y' > 0$, and if $c \in (3, \infty)$, $y' < 0$. Thus, as y is continuous, $c \geq 0$ implies that $y(x)$ must be between c and 3 for all $x \geq 0$.

8
6 points

8. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

8.(a). *(3 points)* Use the Remainder Estimate for the Integral Test to find an upper bound for the error in using S_{10} (the 10th partial sum) to approximate the sum of this series.

SOLUTION: If R_{10} denotes the error described above, the Remainder Estimate for the Integral Test tells us that

$$R_{10} \le \int_{10}^{\infty} \frac{1}{x^4}\, dx = \lim_{c\to\infty} \frac{-3}{x^3}\bigg|_{10}^{c} = \lim_{c\to\infty} \frac{-3}{c^3} - \frac{-3}{10^3} = \frac{3}{1000}.$$

8.(b). *(3 points)* How many terms are required to ensure that the sum is accurate to within 0.1?

SOLUTION: In order to ensure that the error in estimate is less than 0.1 with the Remainder Estimate for the Integral Test, we must solve

$$\int_{n}^{\infty} \frac{1}{x^4}\, dx \le \frac{1}{10}.$$

Following the solution in (a),

$$\int_{n}^{\infty} \frac{1}{x^4}\, dx = \lim_{c\to\infty} \frac{-3}{c^3} - \frac{-3}{n^3} = \frac{3}{n^3},$$

and so we must solve

$$\frac{3}{n^3} \le \frac{1}{10},$$

or $30 \le n^3$, which means that $\sqrt[3]{30} \le n$. As $\sqrt[3]{30} \approx 3.1$, we have that n must be at least 4 in order for S_n to approximate the series within 0.1.

9
12 points

9. Determine whether the series is convergent or divergent and circle the corresponding answer. Then write the test allows one to determine convergence or divergence

9.(a). *(3 points)* $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}}$

convergent divergent

Test: p-series test

9.(b). *(3 points)* $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{n^2 - 3}$

convergent divergent

Test: alternating series test

9.(c). *(3 points)* $\sum_{n=1}^{\infty} \cos\left(\frac{5}{n}\right)$

convergent divergent

Test: test for divergence

9.(d). *(3 points)* $\sum_{n=1}^{\infty} \frac{n^2+5}{(n+2)!}$

convergent divergent

Test: ratio test

10
6 points

10. MULTIPLE CHOICE: Circle the best answer below.

10.(a). *(2 points)* The sequence $a_n = 1 - 0.2^n$

converges to 0. converges, but not to 0. diverges.

10.(b). *(2 points)* The sequence $a_n = \dfrac{3n-4}{2n-1}$

converges to 0. converges, but not to 0. diverges.

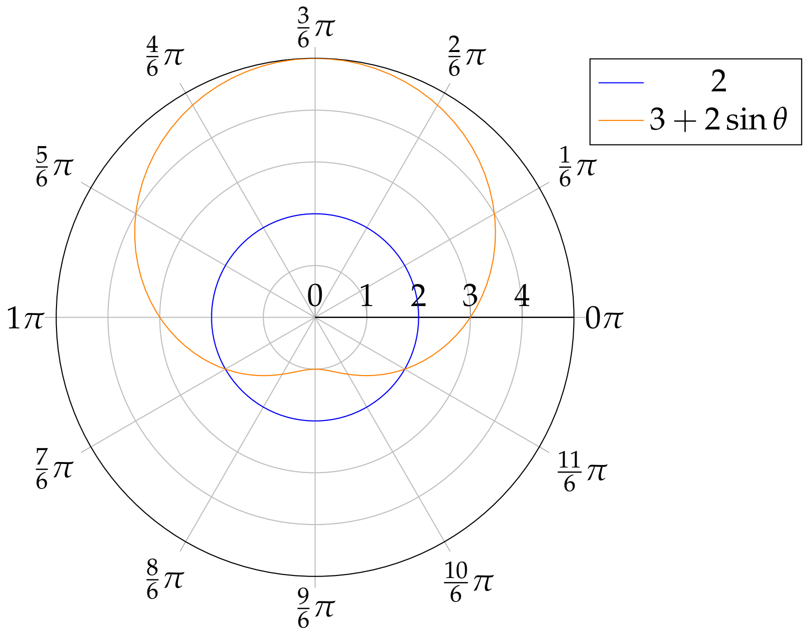
10.(c). *(2 points)* The sequence $a_n = n + \dfrac{1}{n}$

converges to 0. converges, but not to 0. diverges.

11
8 points

11.

11.(a). *(4 points)* Sketch the curves $r = 2$ and $r = 3 + 2\sin\theta$ on the axes below.



11.(b). *(4 points)* Write an integral that represents the area contained outside the first curve ($r = 2$) and inside the second curve ($r = 3 + 2\sin(\theta)$). Do not evaluate the integral.

SOLUTION: From the graph (or via algebraic solution), the bound of integration should be $-\pi/6$ to $7\pi/6$. Thus, the integral is

$$\frac{1}{2}\int_{-\pi/6}^{7\pi/6} (3 + 2\sin(\theta))^2 - 2^2 \, d\theta$$

12
8 points

12. MULTIPLE CHOICE: Circle the best answer below.

12.(a). *(2 points)* Is the following statement ALWAYS, SOMETIMES, or NEVER true?

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

ALWAYS SOMETIMES NEVER

12.(b). *(2 points)* Is the following statement ALWAYS, SOMETIMES, or NEVER true?

If $\sum a_n$ converges, then $\sum |a_n|$ converges.

ALWAYS SOMETIMES NEVER

12.(c). *(2 points)* The graph of $\begin{cases} x = t^2 - 3 \\ y = -t \end{cases}$ for $-\infty < t < \infty$ is a

line parabola circle ellipse

12.(d). *(2 points)* The graph of $\begin{cases} x = t^2 - 3 \\ y = -t^2 \end{cases}$ for $-\infty < t < \infty$ is a

line parabola circle ellipse