Math 2300, Midterm 2

October 23, 2017

PRINT YOUR NAME: ______________________________

PRINT INSTRUCTOR'S NAME: ______________________________

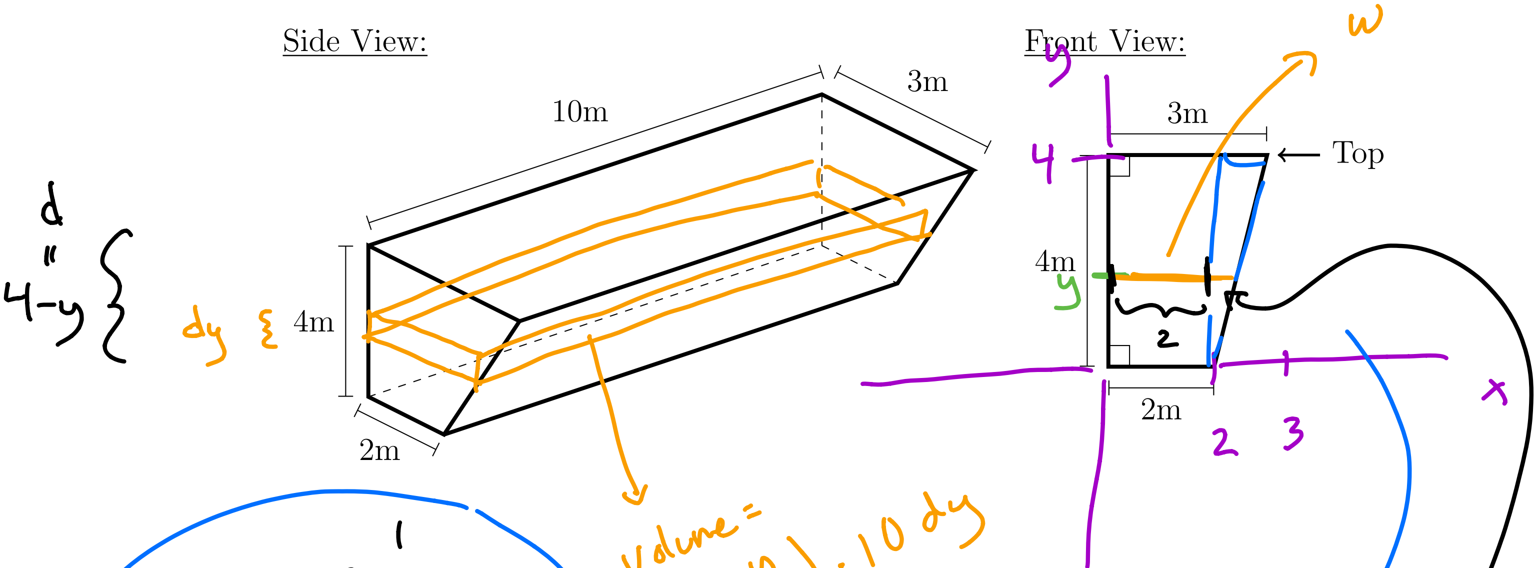
Mark your section/instructor:

☐	Section 001	Brendt Gerics	8:00-8:50
☐	Section 002	Tyler Schrock	8:00-8:50
☐	Section 003	Xingzhou Yang	9:00-9:50
☐	Section 004	Albert Bronstein	9:00-9:50
☐	Section 006	Sebastian Bozlee	10:00-10:50
☐	Section 007	Athena Sparks	11:00 - 11:50
☐	Section 008	Trevor Jack	4:00 - 4:50
☐	Section 009	Jun Hong	12:00 - 12:50
☐	Section 011	Isabel Corona	1:00 - 1:50
☐	Section 012	Hanson Smith	2:00 - 2:50
☐	Section 013	Noah Williams	3:00 - 3:50
☐	Section 014	John Willis	3:00 - 3:50
☐	Section 015	Robert Hines	4:00 - 4:50
☐	Section 016	Sarah Salmon	4:00 - 4:50
☐	Section 017	Xingzhou Yang	8:00 - 8:50
☐	Section 880	Trubee Davison	2:00 - 2:50

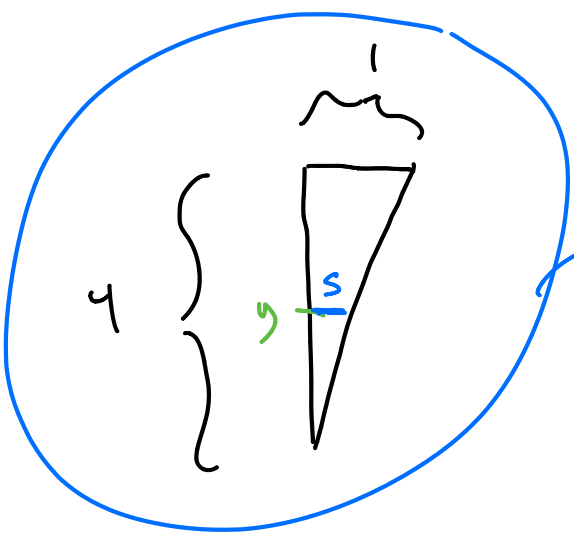
Question	Points	Score
1	10	
2	8	
3	10	
4	8	
5	10	
6	10	
7	10	
8	5	
9	10	
10	7	
11	12	
Total:	100	

- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated. Use full mathematical or English sentences.
- You have 90 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like 100/7 or expressions like ln(3)/2 as is.
- When done, give your exam to your instructor, who will mark your name off on a photo roster.
- We hope you show us your best work!

1. (10 points) Set up, **but do not evaluate**, an integral that represents the work required to empty the tank by pumping all of the water to the top of the tank. You may use $\rho = 1000$ kg/m^3 for the density of water and $g = 9.8$m/sec^2 for the acceleration due to gravity.



volume $= \left(2+\frac{y}{4}\right)\cdot 10\,dy$

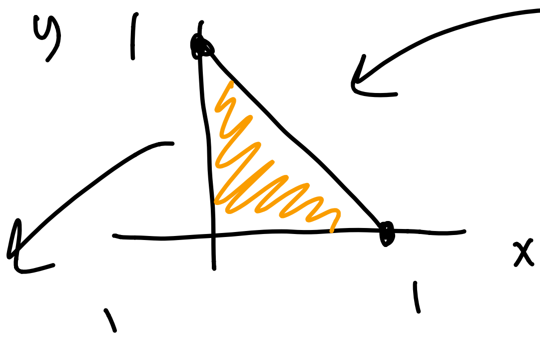


$$\frac{y}{s} = \frac{4}{1} \qquad s = \frac{y}{4}$$

$$w = 2 + \frac{y}{4}$$

$$\text{Work} = \int_{y=0}^{y=4} \rho\cdot g\,(4-y)\left(2+\frac{y}{4}\right)\cdot 10\,dy$$

2. (8 points) Consider a lamina of uniform density bounded by the lines $x + y = 1$, $x = 0$, and $y = 0$. Compute the center of mass $(\bar{x}, \bar{y})$ of the lamina.



$y = 1 - x$

intuitively $\bar{x} = \bar{y}$

Area $= \frac{1}{2}$

$$\bar{x} = \frac{1}{\frac{1}{2}} \int_{x=0}^{x=1} x\, f(x)\, dx$$

$$= 2 \int_0^1 x(1-x)\, dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3} \Big]_0^1 \right)$$

$$= 2\left(\frac{1}{2} - \frac{1}{3} \right) = \frac{2}{6} = \frac{1}{3}$$

$$\bar{y} = \frac{1}{3}$$

3. (10 points) Consider the series $\sum_{n=2}^{\infty} \frac{3^n}{5^{n-1}}$. Determine if the series converges or diverges. If the series converges, find the sum of the series.

For Exam 2, either geometric or telescoping

$$r = \frac{a_{n+1}}{a_n} = \frac{\frac{3^{n+1}}{5^{n+1-1}}}{\frac{3^n}{5^{n-1}}} = \frac{3^{\cancel{n}+1}}{3^{\cancel{n}}} \cdot \frac{5^{\cancel{n}-1}}{5^{\cancel{n}}} = \frac{3}{5}$$

Series will converge since $\left|\frac{3}{5}\right| < 1$.

Q: What does it converge to?

$r = \frac{3}{5}$ $\qquad a =$ "first term" $= \frac{3^2}{5^{2-1}} = \frac{9}{5}$

converges to

$$\frac{a}{1-r} = \frac{\frac{9}{5}}{1-3/5} = \frac{9}{2}$$

4. (a) (5 points) **Multiple Choice.** Suppose that $\sum_{n=1}^{\infty} a_n = 5$ and $\underbrace{s_N = a_1 + a_2 + \cdots + a_N}_{\text{Nth partial sum}}$.

Which **one** of these statements is true?

(I) ~~$\lim_{n\to\infty} a_n = 5$ and $\lim_{N\to\infty} s_N = 0$~~

(II) $\lim_{n\to\infty} a_n = 0$ and $\lim_{N\to\infty} s_N = 0$

(III) ~~$\lim_{n\to\infty} a_n = 5$ and $\lim_{N\to\infty} s_N = 5$~~

(IV) $\lim_{n\to\infty} a_n = 0$ and $\lim_{N\to\infty} s_N = 5$

(V) ~~$\lim_{n\to\infty} a_n$ can not be determined but $\lim_{N\to\infty} s_N = 0$~~

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} s_N$$

$a_n = 1/n^2$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

$b_n = 1/n$ $\quad \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0$$

(b) (3 points) **Multiple Choice.** Suppose that $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = c$.

If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then c could be equal to

(I) 1 (II) 0 (III) 0.5

(IV) ~~−1~~ (V) 2 (VI) ~~−2~~

If $c = 1, 2, .5$, by LCT both series would behave the same way regarding convergence.

For LCT, $c \neq 0$, $c \neq +\infty$

5. (10 points) Determine if $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ converges or diverges. If the series converges, find the sum.

geometric or telescoping

$$\frac{1}{n}-\frac{1}{n+1}=$$

$$\frac{n+1}{n(n+1)}-\frac{n}{n(n+1)}=\frac{1}{n(n+1)}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

this could be harder starting point

partial fractions

$$\frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1} \quad \cdots \quad A=1,\ B=-1$$

$$S_n=\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots$$

$$+\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}$$

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}1-\frac{1}{n+1}=1$$

converges to 1

6. Consider the series $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$.

$$\frac{n}{e^{n^2}} \leq \frac{e^{n^2/2}}{e^{n^2}} = \frac{1}{e^{n^2/2}} \leq \frac{1}{e^{n/2}} = \left(\frac{1}{\sqrt{e}}\right)^n$$

(a) (2 points) Which test will you use to determine the convergence of the series?

Integral Test, or Ratio Test, or DCT

(b) (3 points) Check that the hypothesis of the test are satisfied. If there are no hypotheses for the test you've chosen, state 'none necessary for chosen test'.

$f(x) = \frac{x}{e^{x^2}}$

① f continuous $[1,\infty)$

② f positive on $[1,\infty)$

$\frac{x}{e^{x^2}} > 0$ for $x \in [1,\infty)$

③ Decreasing:

$$f'(x) = \frac{1-2x^2}{e^{x^2}} < 0.$$

if $x \geq 1$

(c) (5 points) Determine if the series converges or diverges.

if have constant that's increasing obviously decreasing so no derivative needed.

The integral test is satisfied.

$$\int_1^{\infty} \frac{x}{e^{x^2}}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{x}{e^{x^2}}\,dx$$

$u = x^2 \ldots$

$$= \lim_{b\to\infty} \left(-\frac{1}{2} e^{-x^2} \Big]_1^b \right)$$

$$= \lim_{b \to \infty} \left(-\frac{1}{2} e^{-b^2} - -\frac{1}{2} e^{-1^2} \right)$$

$$= \frac{1}{2e}$$

Since improper integral converges then the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$ converges also, by the Integral Test

7. Consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$.

(a) (2 points) Which test will you use to determine the convergence of the series?

LCT

(b) (3 points) Check that the hypothesis of the test are satisfied. If there are no hypotheses for the test you've chosen, state 'none necessary for chosen test'.

$a_n = \frac{1}{\sqrt{n^2+1}} > 0$ ✓ $\boxed{b_n = \frac{1}{\sqrt{n^2}} = \frac{1}{n} > 0}$ ✓

I know $\sum \frac{1}{n}$ diverges.

(c) (5 points) Determine if the series converges or diverges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} =$$

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2(1+\frac{1}{n^2})}} = 1$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\sqrt{1+\cancel{\frac{1}{n^2}}^{\to 0}}} = 1 = c$$

Since c is a positive and finite value, LCT is satisfied.

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges.

More pitfalls:

- By LCT, $\cancel{\frac{1}{\sqrt{n^2+1}}}$ ~~diverges~~.

  remember

  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges

doesn't make sense - saying a series is a term

bad explanation!

- $\cancel{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}} = \frac{1}{n}}$ ~~diverges~~.

Scrap: answer to question

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

$$a_n \approx c\, b_n$$

$$\sum_{n=1}^{\infty} a_n \approx \sum_{n=1}^{\infty} c\, b_n$$

converges

converges

Show absolute convergence

$\sum_{n=1}^{\infty} \frac{1}{(n+4)^2}$ converges by LCT or DCT

8. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+4)^2}$.

(a) (2 points) Show that this series converges using an appropriate test.

Use AST. $a_n = \frac{1}{(n+4)^2}$

① • $\lim_{n\to\infty} \frac{1}{(n+4)^2} = 0$ ✓

② • a_n's decreasing ✓

(b) (3 points) Estimate the error in using the the 95th partial sum

$$s_{95} = \sum_{n=1}^{95} \frac{(-1)^{n+1}}{(n+4)^2} = S$$

to approximate the sum of the infinite series.

$$\underbrace{|S - S_{95}|}_{R_{95}} \leq a_{95+1} = a_{96}$$

$$a_{96} = \frac{1}{(96+4)^2} = \frac{1}{100^2}$$

9. (10 points) Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n n!}{(2n)!}$ converges absolutely, converges but not absolutely, or diverges.

conditional convergence

order to do this problem:

① First check absolute convergence.

② If does not converge absolutely, then check conditional convergence.

③ If does does not converge conditionally, then diverges.

Ratio Test always tests for absolute convergence. B/c of absolute values.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{\cancel{(-1)^{n+1}}\, 3^{n+1}\,(n+1)!}{(2(n+1))!}\right|}{\left|\frac{\cancel{(-1)^{n}}\, 3^{n}\, n!}{(2n)!}\right|}$$

$$= \lim_{n\to\infty} \frac{\cancel{3^{n}}^{+1}}{\cancel{3^n}} \cdot \underbrace{\frac{(n+1)!}{n!}}_{\frac{n+1}{1}} \cdot \underbrace{\frac{(2n)!}{(2n+2)!}}_{\frac{\cancel{(2n)}\cancel{(2n-1)}\cancel{(2n-2)}\cdots\cancel{2\cdot 1}}{(2n+2)(2n+1)\cancel{(2n)}\cdots\cancel{2\cdot 1}}}$$

$$= \lim_{n\to\infty} \frac{3(n+1)}{(2n+2)(2n+1)} = 0 < 1$$

By Ratio Test, converges absolutely.

10. Determine if the following statements are always true, sometimes true, or never true.

(a) (3 points) **Multiple choice.** If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(I) Sometimes (II) Always (III) Never

(b) (4 points) **Multiple choice.** Let c_n be a sequence satisfying $\frac{1}{n^2} < c_n < \frac{1}{n}$ for $n \geq 1$. Then $\sum_{n=1}^{\infty} c_n$ converges:

(I) Sometimes (II) Always (III) Never

$c_n = \frac{2}{n^2}$ $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges

$c_n = \frac{1}{2n}$ $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges

11. **Multiple Choice.** For each of the following sequences and series, determine the appropriate response.

(a) (2 points) $a_n = \frac{n}{n+1}$

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

(I) Converges (II) Diverges

(b) (2 points) $c_n = (-1)^n \sqrt{n}$

(I) Converges (II) Diverges

(c) (2 points) $\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Test for Div.

$$\lim_{n\to\infty} \frac{n-1}{2n+1} = \frac{1}{2} \neq 0$$

(I) Converges (II) Diverges

(d) (2 points) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{3}\right)^n$

(I) Converges (II) Diverges

$$\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^n \quad r = -\frac{1}{3}$$

$$\left|-\frac{1}{3}\right| < 1$$

(e) (2 points) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{3n}$

AST

(I) Converges (II) Diverges

(f) (2 points) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ $\quad p = 1/2 < 1$

(I) Converges (II) Diverges

$$\sum_{n=1}^{\infty}(-1)^n \frac{1}{3n}$$

$$a_n = \frac{1}{3n}$$

- $\lim_{n\to\infty} \frac{1}{3n} = 0$ ✓

- a_n's decreasing