SEQUENCES AND SERIES EXPECTATIONS

This document summarizes common expectations for the solution of these problems.

In general:

(1) Our students should write enough so that a grader can follow their reasoning process using solely what you have written on the paper.
(2) You do not need to prove any of the common theorems discussed in class, the textbook, projects, or activities.
(3) You must check each hypothesis of any theorem you use. When a hypothesis takes no work to check, it is enough just to acknowledge it. For example: if you need to check that x^2 is continuous, it's enough to write "x^2 is continuous."
(4) When drawing a conclusion using a theorem, you must use the name of the theorem. For example, "Since $\lim_{n\to\infty} n^2 = \infty$, the series $\sum_{n=1}^{\infty} n^2$ diverges by the **test for divergence**."
(5) When saying that something converges or diverges, you must be specific to what exactly converges or diverges. Do not say "**it** converges by the p test," you should write "$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test."

There are two goals: first, for you to learn how to write clear logical arguments, and second, to encourage unambiguous written communication.

The rest of this document consists of the details of the expectations for particular tests or checks.

1. Sequences

1.1. L'Hôpital's Rule.

Using L'Hôpital's Rule should look like this:

$$\lim_{n\to\infty} \frac{n-3}{2^n} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{1}{2^n \ln(2)} = 0.$$

That is, you should indicate that you are using L'Hôpital's Rule by an LH over the equals sign. Of course, L'Hôpital's Rule only applies to $\frac{\infty}{\infty}$ or $\frac{0}{0}$ indeterminate forms whose numerators and denominators you can differentiate, but we trust that you know this from Calculus 1. It is **not** necessary for you to explicitly replace the limit of the sequence with the limit of a function defined on $[1,\infty)$ before applying L'Hôpital's Rule.

1.2. Limits of Rational Functions.

If $f(n) = a_m n^m + a_{m-1} n^{m-1} + \cdots + a_0$ and $g(n) = b_k n^k + b_{k-1} n^{k-1} + \cdots + b_0$ are polynomials of degree m and k respectively, then

$$\lim_{n\to\infty} \frac{f(x)}{g(x)} = \begin{cases} \operatorname{sign}(\frac{a_m}{b_k})\infty & \text{if } m > k \\ \frac{a_m}{b_k} & \text{if } m = k \\ 0 & \text{if } m < k \end{cases}$$

While it may bear a reminder, this rule is considered something that it is expected knowledge from Calculus I. You may therefore use it to compute limits of rational functions immediately, without showing work, and even without foiling. For example,

$$\lim_{n\to\infty} \frac{(3n+4)(2n-3)(n-5)}{(n+1)^3+5n-n^2} = 6$$

is perfectly acceptable.

On the other hand, similar limits involving roots require more work to be shown. For example, in

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1,$$

the middle steps should not be avoided.

1.3. **"Faster/Slower" functions.** If $\{a_n\}$ and $\{b_n\}$ are strictly positive sequences, we often express that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ or $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = 0$ by saying that "a_n grows faster than b_n" or "b_n grows slower than a_n." Many common limits of sequences can be worked out just by knowing which functions grow faster than others:

b_n	grows slower than	a_n
$\ln(n)$		any polynomial
any polynomial of degree k		any polynomial of degree greater than k
any polynomial		any growing exponential (a^n, where $a > 1$)
b^n		a^n, where $a > b > 1$
any growing exponential		$n!$
$n!$		n^n

These are the only comparisons of growth rates that you are expected to know and the only ones you may use without further proof. For example,

$$\boxed{\lim_{n\to\infty} \frac{5n^3+2n-4}{3^n} = 0, \text{ since 3 is larger than 1 and exponentials grow faster than polynomials}}$$

is acceptable.

1.4. **Increasing/decreasing sequences.** There are three ways to show that a sequence is increasing (decreasing):

(1) directly, checking algebraically that $a_{n+1} \geq a_n$ ($a_{n+1} \leq a_n$)
(2) via the derivative (i.e. checking algebraically that $f'(x) \leq 0$ for $x \geq 1$ where $f(n) = a_n$ for all $n \in \mathbb{Z}_{\geq 1}$)
(3) easy to derive from previously basic fact, which includes:
   - $(\ln x)^r$, x^r and e^{rx} are increasing, positive and unbounded in $(1, \infty)$, for any positive real number r. It follows that the sequences $\{\frac{1}{(\ln n)^r}\}_{n=1}^\infty$, $\{\frac{1}{n^r}\}_{n=1}^\infty$ and $\{\frac{1}{e^{rn}}\}_{n=1}^\infty$ are positive and decreasing for $n \geq 1$ and tend to 0 as n tends to ∞, hence statisfy hypothesis of the alternating series tests.
   - The product of increasing (decreasing) sequences is increasing (decreasing).
   - The sum of increasing (decreasing) sequences is increasing (decreasing).

- Compositions work as follows:

f	g	then	$f \circ g$
increasing on $\mathbb{R}$	increasing on $\mathbb{R}$		increasing on $\mathbb{R}$
increasing on $\mathbb{R}$	decreasing on $\mathbb{R}$		decreasing on $\mathbb{R}$
decreasing on $\mathbb{R}$	increasing on $\mathbb{R}$		decreasing on $\mathbb{R}$
decreasing on $\mathbb{R}$	decreasing on $\mathbb{R}$		increasing on $\mathbb{R}$

The most common sequences that do not qualify belong (3) are quotients of increasing divergent sequences. An example is $\frac{(\ln n)^9}{n^{1/9}}$ is eventually decreasing, but first increases for quite a "while". For instance, proving that this sequence satisfies the hypothesis of the Alternating Series Test may include using several iterations of L'Hôpital's rule, and certainly includes computing the sign of $f'(x)$.

2. Series

2.1. **Telescoping Series.** A telescoping series problem should look like

$$\begin{aligned}
\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) &= \sum_{n=1}^{\infty} [\ln(n+1) - \ln(n)] \\
s_n &= (\ln(2) - \ln(1)) + (\ln(3) - \ln(2)) + \cdots + (\ln(n+1) - \ln(n)) \\
&= \ln(n+1) - \ln(1) \\
\lim_{n\to\infty} \ln(n+1) - \ln(1) &= \infty. \\
\text{Therefore } \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) &\text{ diverges.}
\end{aligned}$$

That is, at some point you must write out the general form of the n^{th} partial sum, then take its limit before reaching a conclusion. It is not enough to write out the first few partial sums without writing down the general form.

It is not enough to write out the series as an infinite sum and perform the cancellation there, as this gives incorrect results for divergent telescoping series. For example,

$$\begin{aligned}
\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) &= \sum_{n=1}^{\infty} [\ln(n+1) - \ln(n)] \\
&= (\ln(2) - \ln(1)) + (\ln(3) - \ln(2)) + (\ln(4) - \ln(3)) + \cdots \\
&= -\ln(1) + (\ln(2) - \ln(2)) + (\ln(3) - \ln(3)) + (\ln(4) - \ln(4)) + \cdots \\
&= 0
\end{aligned}$$

is incorrect.

2.2. p**-Test.** A use of the p-series test should look like

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} \text{ converges by the } p\text{-series test (optional: with p=(3/2) > 1).}$$

That is, the name of the theorem and the correct conclusion must appear.

2.3. **Test for Divergence.** A test for divergence problem should look like

$$\lim_{n\to\infty} \frac{n+1}{n+2} = 1 \text{ (optional: } \lim_{n\to\infty} \cdots = 1 \neq 0).$$

$$\text{Therefore } \sum_{n=1}^{\infty} \frac{n+1}{n+2} \text{ diverges by the divergence test.}$$

That is, you must explicitly take the limit of the terms, then conclude with the name of the theorem.

2.4. **Geometric Series Test.** A use of the Geometric Series Test should look like

$$\sum_{n=1}^{\infty} 2\left(\frac{3}{5}\right)^{n+2} \quad \text{converges by the geometric series test since } |r| = \frac{3}{5} < 1.$$

or

$$\sum_{n=1}^{\infty} 2\left(\frac{3}{5}\right)^{n+2} = \sum_{n=1}^{\infty} 2\left(\frac{3}{5}\right)^{3}\left(\frac{3}{5}\right)^{n-1} \quad \text{converges by the geometric series test.}$$

That is, you must either explicitly identify r and compare its absolute value with 1, or you must rewrite the series in the standard form $\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=0}^{\infty} ar^{n}$ before concluding correctly with the geometric series test. It is not necessary to explicitly note that the series is geometric. Writing "$r < 1$" instead of "$|r| < 1$" is incorrect.

2.5. **Integral Test.** An Tntegral Test problem should look like

$$\frac{1}{(x+1)\ln(x+1)} \text{ is continuous, decreasing, and positive}$$

$$\begin{aligned}
\int_1^{\infty} \frac{1}{(x+1)\ln(x+1)}\,dx &= \lim_{t\to\infty} \int_1^{t} \frac{1}{(x+1)\ln(x+1)}\,dx \\
&= \lim_{t\to\infty} \int_2^{t+1} \frac{1}{u}\,du, \quad u = \ln(x+1),\ du = \frac{1}{x+1}dx \\
&= \lim_{t\to\infty} \ln(u)\,\Big|_2^{t+1} \\
&= \lim_{t\to\infty} \ln(t+1) - \ln(2) \\
&= \infty
\end{aligned}$$

By the Integral Test,

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)} \text{is divergent.}$$

The three hypotheses that $f(x)$ is decreasing, continuous, and positive must be addressed explicitly, theTintegral's convergence or divergence must be worked out in full, you must state in addition that the integral is convergent or divergent, and you must state the conclusion with the name of the theorem.

2.6. **Comparison Test.** A comparison test problem should look like

$$\frac{n^2+3}{n^3-\sqrt{n}} \geq \frac{n^2}{n^3-\sqrt{n}} \geq \frac{n^2}{n^3} = \frac{1}{n} \geq 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges by the } p\text{-test.}$$

$$\text{Therefore } \sum_{n=1}^{\infty} \frac{n^2+3}{n^3-\sqrt{n}} \text{ diverges by the comparison test.}$$

That is, you must explicitly write (and justify if not obvious) that $0 \leq a_n \leq b_n$, justify that the series being compared with is convergent/divergent, then conclude with the name of the theorem. It is not necessary to restate the hypotheses in the conclusion statement.

2.7. **Limit Comparison Test.** A use of the limit comparison test should look like

$$0 \leq \frac{\arctan(n)}{n^2+1}, \frac{1}{n^2}.$$

$$\lim_{n\to\infty} \frac{\arctan(n)/(n^2+1)}{1/n^2} = \left(\lim_{n\to\infty} \frac{n^2}{n^2+1}\right) \cdot \left(\lim_{n\to\infty} \arctan(n)\right) = \frac{\pi}{2}.$$

$$0 < \frac{\pi}{2} < \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges by the } p\text{-test.}$$

$$\text{Therefore } \sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2+1} \text{ converges by the limit comparison test.}$$

The non-negativity of the terms, the limit, the check that $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, and a conclusion with the theorem name are all necessary.

If a_n, b_n are only eventually positive, awarness of this fact must be shown explicitly: ($0 < a_n, b_n$ "is eventually true" or "is true for n large enough").

2.8. **Alternating Series Test.** A use of the alternating series test should look like

$$\frac{1}{\sqrt{n}} \text{ is decreasing}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

$$\text{Therefore } \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ converges by the alternating series test.}$$

You must check that a_n is decreasing, that $\lim_{n\to\infty} a_n = 0$, and conclude with the name of the theorem.

2.9. **Ratio Test.** A use of the ratio test should look like

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^2+3(n+1)+4}\cdot\frac{n^2+3n+4}{2^n}\right| &= \lim_{n\to\infty}\left|\frac{2(n^2+3n+4)}{(n+1)^2+3(n+1)+4}\right| \\ &= 2\end{aligned}$$

By the ratio test, $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^2+3n+4}$ diverges.

You must take the limit of the ratios, and draw the appropriate conclusion while citing the ratio test. by $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.