1(a) $\int \sin x \cos x \, dx$

$u = \sin x \qquad dv = \cos x \, dx$

$du = \cos x \, dx \qquad v = \sin x$

$\int \sin x \cos x \, dx = \sin^2 x - \int \sin x \cos x \, dx$

$2\int \sin x \cos x \, dx = \sin^2 x$

$\int \sin x \cos x \, dx = \frac{1}{2}\sin^2 x + C$

1(b) $\int \sin x \cos x \, dx \qquad u = \cos x \quad dv = \sin x \, dx$

$du = -\sin x \, dx \quad v = -\cos x$

$\int \sin x \cos x \, dx = -\cos^2 x - \int \sin x \cos x \, dx$

$2\int \sin x \cos x \, dx = -\cos^2 x + C_1 = \sin^2 x - 1 + C_1$

call it C

$\int \sin x \cos x \, dx = \frac{1}{2}\sin^2 x + \frac{(C_1 - 1)}{2} =$

$\frac{1}{2}\sin^2 x + C$

1(c) $w = \sin x \quad dw = \cos x\,dx$

$$\int w\,dw = \frac{1}{2}w^2 + C$$

Answer $\frac{1}{2}\sin^2 x + C$

1(d) $w = \cos x \quad dw = -\sin x\,dx$

$-dw = \sin x\,dx$

$$-\int w\,dw = -\frac{1}{2}w^2$$

$-\frac{1}{2}\cos^2 x$ but $\frac{1}{2}\sin^2 x + \frac{1}{2}\cos^2 x = 1/2$

$-\frac{1}{2}\cos^2 x = \frac{1}{2}\sin^2 x - \frac{1}{2}$

$\frac{1}{2}\sin^2 x \underbrace{- \frac{1}{2} + C}$

call it C

Answer $\frac{1}{2}\sin^2 x + C$

(e) $\sin 2x = 2\sin x\cos x$

$\frac{1}{2}\sin 2x = \sin x\cos x$

$\int \sin x\cos x\,dx = \frac{1}{2}\int \sin(2x)\,dx =$

$-\frac{1}{4}\cos(2x) = -\frac{1}{4}\left[1-2\sin^2 x\right] =$

$-\frac{1}{4} + \frac{1}{2}\sin^2 x + C_1$

Let $C = C_1 - \frac{1}{4}$

Answer $\frac{1}{2}\sin^2 x + C$

1(f) See Above

$\boxed{2}$ $\int_0^1 f(e^x)e^x dx$

$u=e^x \quad du=e^x dx$

$x=0 \quad u=e^0=1$

$x=1 \quad u=e^1=e$

$$\int_0^1 f(e^x)e^x dx=\int_1^e f(u)du$$

VARIABLE u CAN be replaced by Any other VARIABLE including x

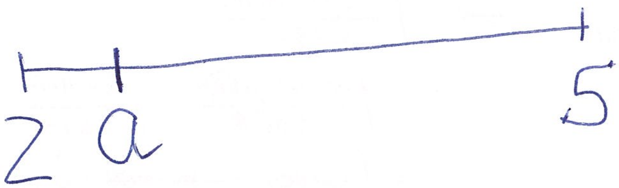
So $$\int_0^1 f(e^x)e^x dx=\int_1^e f(x)dx$$

③ ⓐ $\displaystyle\int_2^5 \frac{x\,dx}{\sqrt{x^2-4}}$ is improper

because $y = \dfrac{x}{\sqrt{x^2-4}}$ is not

defined at $x=2$

ⓑ



$$\int_2^5 \frac{x}{\sqrt{x^2-4}}\,dx = \lim_{a\to 2^+} \int_a^5 \frac{x}{\sqrt{x^2-4}}\,dx$$

$$\int \frac{x}{\sqrt{x^2-4}}\,dx \qquad u = x^2-4 \qquad \tfrac{1}{2}du = x\,dx$$
$$du = 2x\,dx$$

$$\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot\frac{u^{1/2}}{1/2} = u^{1/2} = \sqrt{u}$$

$$\lim_{a\to 2^+} \int_a^5 \frac{x}{\sqrt{x^2-4}}\,dx = \lim_{a\to 2^+} \sqrt{x^2-4}\,\Big|_a^5 =$$

3 b continued.

$$\lim_{a \to 9^+} \left[\sqrt{21} - \sqrt{a^2-4}\right] = \sqrt{21}$$

#4 $\int_0^1 f(t)\,dt = 5$

a $\int_0^{0.5} f(2t)\,dt$ $\quad u = 2t \quad \frac{1}{2}du = dt$

$du = 2dt$

$t = 0 \quad u = 0$

$t = 0.5 \quad u = 1$

$$\int_0^{0.5} f(2t)\,dt = \frac{1}{2}\int_0^1 f(u)\,du = \frac{1}{2}\cdot 5 = 5/2$$

b $\int_0^1 f(1-t)\,dt$ $\quad u = 1-t \quad -du = dt$

$du = -dt$

$t = 0 \quad u = 1$

$t = 1 \quad u = 0$

$$\int_0^1 f(1-t)\,dt = -\int_1^0 f(u)\,du$$

4(b) continued

$$\int_0^1 f(1-t)\,dt = \int_0^1 f(u)\,du = \underline{5}$$

4(c) $\int_1^{1.5} f(3-2t)\,dt$

$u = 3-2t \quad du = -2dt \quad -\frac{1}{2}du = dt$

$t = 1 \quad u = 1$

$t = 1.5 \quad u = 0$

$$\int_1^{1.5} f(3-2t)\,dt = -\frac{1}{2}\int_1^0 f(u)\,du =$$

$$\frac{1}{2}\int_0^1 f(u)\,du = \underline{5/2}$$

5ⓐ $\int 2x \cos(x^2)dx$

u-sub $\quad u = x^2 \quad du = 2x\,dx$

$$\int \cos(u)\,du = \sin(u) + C$$

$$\int 2x\cos(x^2)dx = \sin(x^2) + C$$

5ⓑ $\int e^{2x}\sin(2x)dx$ by parts

$u = e^{2x} \quad dv = \sin(2x)dx$

$du = 2e^{2x}dx \quad v = -\frac{1}{2}\cos(2x)$

$$\int e^{2x}\sin(2x)dx = -\frac{1}{2}e^{2x}\cos(2x) + \int e^{2x}\cos(2x)dx$$

Apply by parts to $\int e^{2x}\cos(2x)dx$

$u = e^{2x} \quad dv = \cos(2x)dx$

$du = 2e^{2x}dx \quad v = \frac{1}{2}\sin(2x)$

5(b) continued

$$\int e^{2x}\cos(2x)dx = \frac{1}{2}e^{2x}\sin(2x) - \int e^{2x}\sin(2x)dx$$

$$\int e^{2x}\sin(2x)dx = -\frac{1}{2}e^{2x}\cos(2x) + \int e^{2x}\cos(2x)dx =$$

$$-\frac{1}{2}e^{2x}\cos(2x) + \frac{1}{2}e^{2x}\sin(2x) - \int e^{2x}\sin(2x)dx$$

$$2\int e^{2x}\sin(2x)dx = -\frac{1}{2}e^{2x}\cos(2x) + \frac{1}{2}e^{2x}\sin(2x)$$

$$\int e^{2x}\sin(2x)dx = -\frac{1}{4}e^{2x}\cos(2x) + \frac{1}{4}e^{2x}\sin(2x) + C$$

5(c) $\int \cos^2\theta d\theta$ $\quad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$\frac{1}{2}\int (1+\cos 2\theta)d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)$$

$\sin(2\theta) = 2\sin\theta\cos\theta$

5(c) continued

$$\int \cos^2\theta \, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) =$$

$$\frac{1}{2}\theta + \frac{1}{4}2\sin\theta\cos\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

CAN ALSO use integration by parts

$$\int \cos^2\theta \, d\theta = \int \cos\theta\cos\theta \, d\theta$$

$u = \cos\theta \quad dv = \cos\theta \, d\theta$

$du = -\sin\theta \, d\theta \quad v = \sin\theta$

$$\int \cos^2\theta \, d\theta = \sin\theta\cos\theta + \int \sin^2\theta \, d\theta$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\int \cos^2\theta \, d\theta = \sin\theta\cos\theta + \int (1 - \cos^2\theta) \, d\theta =$$

$$\sin\theta\cos\theta + \theta - \int \cos^2\theta \, d\theta$$

5(c) continued

$$2\int \cos^2\theta \, d\theta = \sin\theta\cos\theta + \theta$$

$$\int \cos^2\theta \, d\theta = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

5(d) $\int x^2 \sin(x)\,dx$ $\quad u = x^2 \quad dv = \sin x\,dx$

$du = 2x\,dx \quad v = -\cos x$

$$= -x^2\cos(x) + 2\int x\cos x\,dx$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x$$

$u = x \quad dv = \cos x\,dx \quad du = 1dx \quad v = \sin x$

$$\int x^2\sin(x)\,dx = -x^2\cos(x) + 2[x\sin x + \cos x]$$

$$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$

5(e) $\int \frac{1}{x^2\sqrt{16-x^2}}\,dx$ trig sub

$x = 4\sin\theta \quad 16-x^2 = 16-16\sin^2\theta = 16\cos^2\theta$

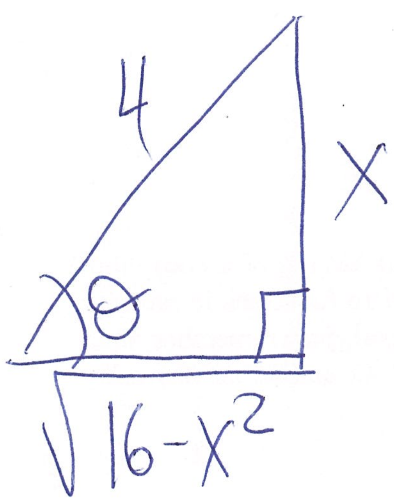
$dx = 4\cos\theta\, d\theta$

$$4\int \frac{\cos\theta}{16\sin^2\theta\sqrt{16\cos^2\theta}}\,d\theta = \frac{1}{16}\int \frac{1}{\sin^2\theta}\,d\theta =$$

$$\frac{1}{16}\int \csc^2\theta\, d\theta = -\frac{1}{16}\cot\theta + C$$

$\sin\theta = x/4$

$$\int \frac{1}{x^2\sqrt{16-x^2}}\,dx =$$



$$-\frac{1}{16}\,\frac{\sqrt{16-x^2}}{x} + C$$

5(f) $\dfrac{3x^2+6}{x^2(x^2+3)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+3}$

$3x^2+6 = Ax(x^2+3) + B(x^2+3) + (Cx+D)x^2$

$\underline{x=0} \Rightarrow \quad 6 = 3B \qquad B = 2$

$3x^2+6 = Ax^3 + 3Ax + 2x^2 + 6 + Cx^3 + Dx^2$

$x^3: \quad 0 = A + C \quad$ so $\quad C = -A$

$x^2: \quad 3 = 2 + D \quad$ so $\quad D = 1$

$x: \quad 0 = 3A \quad$ so $\quad A = 0$ and $C = 0$

$\displaystyle\int \frac{3x^2+6}{x^2(x^2+3)}\,dx = \int \frac{2}{x^2}\,dx + \int \frac{1}{x^2+3}\,dx =$

$\displaystyle\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\,\text{Arctan}\left(\frac{u}{a}\right)$ $\qquad a^2 = 3$, $a = \sqrt{3}$

$-\dfrac{2}{x} + \dfrac{1}{\sqrt{3}}\,\text{Arctan}\left(\dfrac{x}{\sqrt{3}}\right) + C$

5 (g) $\int \sqrt{25-x^2}\,dx$

$x = 5\sin\theta$

$dx = 5\cos\theta\, d\theta$

$25 - x^2 = 25 - 25\sin^2\theta =$

$25\cos^2\theta$

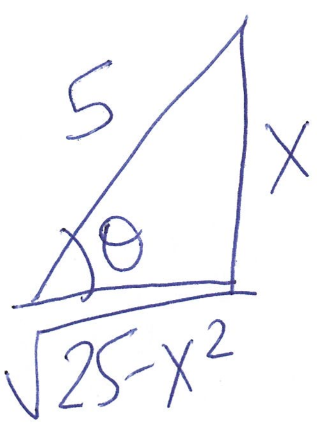
$\int \sqrt{25\cos^2\theta}\; 5\cos\theta\, d\theta$

$= 25\int \cos^2\theta\, d\theta = 25\,\frac{1}{2}\int (1+\cos(2\theta))\,d\theta =$

$\frac{25}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] =$

$\frac{25}{2}\left[\theta + \sin\theta\cos\theta\right]$

$\sin\theta = x/5 \quad \theta = \arcsin(x/5)$



Answer:

$\frac{25}{2}\left[\arcsin\left(\frac{x}{5}\right) + \frac{x}{5}\cdot\frac{\sqrt{25-x^2}}{5}\right] + C$

$= \frac{1}{2}\left[25\arcsin\left(\frac{x}{5}\right) + x\sqrt{25-x^2}\right] + C$

5 h) $\dfrac{3x-1}{x^2-5x+6} = \dfrac{3x-1}{(x-3)(x-2)} = \dfrac{A}{x-3} + \dfrac{B}{x-2}$

$3x-1 = A(x-2) + B(x-3)$

$x=2 \quad 5 = -B \quad \boxed{B = -5}$

$x=3 \quad \boxed{8 = A}$

$$\int \frac{3x-1}{x^2-5x+6}\,dx = \int \frac{3x-1}{(x-3)(x-2)}\,dx =$$

$$8\int \frac{1}{x-3}\,dx - 5\int \frac{1}{x-2}\,dx =$$

$$8\ln|x-3| - 5\ln|x-2| + C =$$

$$\ln\left|\frac{(x-3)^8}{(x-2)^5}\right| + C$$

5 (i) $u = \sin(5x)$ $du = 5\cos(5x)$

$\frac{1}{5}du = \cos(5x)dx$

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5 (j) $\int_2^3 \frac{x^2}{1+x^3}dx$ $u = 1+x^3$

$du = 3x^2 dx$

$\frac{1}{3}du = x^2 dx$

$x=2$ $u=9$

$x=3$ $u=28$ $\frac{1}{3}\int_9^{28} \frac{1}{u}du =$

$\frac{1}{3}\ln u \Big|_9^{28} = \frac{1}{3}[\ln 28 - \ln 9] =$

$\frac{1}{3}\ln[28/9]$

$u = x^2$ $x=0$ $u=0$

5 (k) $\frac{1}{2}du = xdx$ $x=3$ $u=9$

$\frac{1}{2}\int_0^9 e^u du = \frac{1}{2}e^u\Big|_0^9 = \frac{1}{2}[e^9 - 1]$

5(l) $\int x^7 e^{x^4} dx = \int x^4 e^{x^4} \cdot x^3 dx$

$y = x^4 \quad dy = 4x^3 dx \quad \frac{1}{4} dy = x^3 dx$

$\frac{1}{4}\int y e^y dy \quad u = y \quad dv = e^y dy$

$du = 1 dy \quad v = e^y$

$y e^y - \int e^y dy = y e^y - e^y$

$\int x^7 e^{x^4} dx = \frac{1}{4}\left[x^4 e^{x^4} - e^{x^4}\right] + C$

5(m) $\int (\ln x)^2 dx \quad u = (\ln x)^2 \quad dv = 1dx$

$du = 2(\ln x)\frac{1}{x} dx \quad v = x$

$x(\ln x)^2 - 2\int \ln x \, dx =$

$\int \ln x \, dx$

$u = \ln x \quad dv = 1dx$

$du = \frac{1}{x} dx \quad v = x$

$x \ln x - \int 1 dx = x \ln x - x$

$x(\ln x)^2 - 2(x \ln x - x) =$

$x(\ln x)^2 - 2x\ln x + 2x + C$

6 (a) $\int y\sqrt{y^2+1}\,dy \quad u=y^2+1 \quad du=2y\,dy \quad \frac{1}{2}du=y\,dy$

$\frac{1}{2}\int u^{1/2}du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} = \frac{1}{3}u^{3/2}$

$\int y\sqrt{y^2+1}\,dy = \frac{1}{3}(y^2+1)^{3/2} + C$

Also can use trig sub

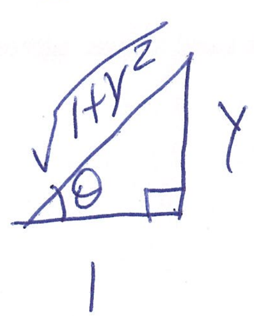
$y=\tan\theta \quad y^2+1=\sec^2\theta \quad dy=\sec^2\theta\,d\theta$

$\int \tan\theta\sqrt{\sec^2\theta}\,\sec^2\theta\,d\theta = \int \tan\theta\sec^3\theta\,d\theta =$

$\int \sec\theta\tan\theta\sec^2\theta\,d\theta \quad u=\sec\theta \quad du=\sec\theta\tan\theta\,d\theta$

$\int u^2du = \frac{1}{3}u^3 \rightarrow \frac{1}{3}\sec^3\theta$

$y=\tan\theta$



$\sec\theta=\sqrt{y^2+1}$

Answer $\frac{1}{3}\left(\sqrt{y^2+1}\right)^3 = \frac{1}{3}(y^2+1)^{3/2}+C$

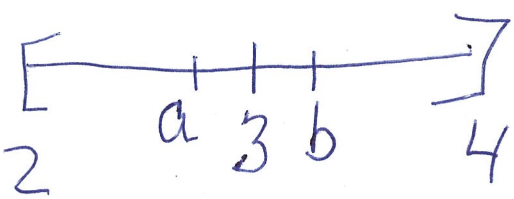
6(b) $\int y\sqrt{y+1}\,dy \qquad u=y+1 \quad du=dy$

$\qquad\qquad\qquad\qquad y=u-1$

$$\int (u-1)u^{1/2}\,du = \int \left(u^{3/2}-u^{1/2}\right)du = \frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}$$

$$\frac{2}{5}(y+1)^{5/2}-\frac{2}{3}(y+1)^{3/2}+C$$

(7a) $$\int_2^4 \frac{dx}{(x-3)^2} = \lim_{a\to 3^-}\int_2^a \frac{1}{(x-3)^2}dx +$$



$$\lim_{b\to 3^+}\int_b^4 \frac{1}{(x-3)^2}dx =$$

$$\lim_{a\to 3^-} \left.\frac{-1}{x-3}\right|_2^a + \lim_{b\to 3^+} \left.\frac{-1}{x-3}\right|_b^4 =$$

$$\lim_{a\to 3^-}\Big[\underbrace{\frac{-1}{a-3}}_{\to\infty}+\frac{1}{-1}\Big] + \lim_{b\to 3^+}\Big[-1+\frac{1}{b-3}\Big] \to \infty$$

$$\int_2^4 \frac{dx}{(x-3)^2} \text{ Diverges}$$

7ⓑ $\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1} dx = \int_{-\infty}^{0} \frac{e^x}{e^{2x}+1} dx +$

$\int_{0}^{\infty} \frac{e^x}{e^{2x}+1} dx = \lim_{a \to -\infty} \int_{a}^{0} \frac{e^x}{e^{2x}+1} dx +$

$\lim_{b \to \infty} \int_{0}^{b} \frac{e^x}{e^{2x}+1} dx$

$u = e^x \quad du = e^x dx$

$\int \frac{e^x}{e^{2x}+1} dx \to \int \frac{1}{u^2+1} du =$

$\text{Arctan}(u)$

$\text{Arctan}(e^x)$

$\lim_{a \to -\infty} \text{Arctan}(e^x) \Big|_a^0 +$

$\lim_{b \to \infty} \text{Arctan}(e^x) \Big|_0^b =$

$\lim_{a \to -\infty} \left[\text{Arctan}\, 1 - \text{Arctan}\, e^a\right] +$

$\lim_{b \to \infty} \left[\text{Arctan}\,(e^b) - \text{Arctan}\, 1\right] =$

$\frac{\pi}{4} - 0 + \frac{\pi}{2} - \pi/4 = \pi/2$

8 a $\int \frac{1}{x^4-5x^2+4}\,dx = \int \frac{1}{(x^2-1)(x^2-4)}\,dx =$

$\int \frac{1}{(x-1)(x+1)(x-2)(x+2)}\,dx$

$\frac{1}{(x-1)(x+1)(x-2)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2} + \frac{D}{x+2}$

$1 = A(x+1)(x-2)(x+2) + B(x-1)(x-2)(x+2) + C(x-1)(x+1)(x+2) +$
$D(x-1)(x+1)(x-2)$

$\underline{x=1}$: $1 = A(-6)$ $A = -1/6$

$\underline{x=-1}$: $1 = B\cdot 6$ $B = 1/6$

$\underline{x=2}$: $1 = 12C$ $C = 1/12$

$\underline{x=-2}$: $1 = -12D$ $D = -1/12$

$\int \frac{1}{x^4-5x^2+4}\,dx = \int \frac{1}{(x-1)(x+1)(x-2)(x+2)}\,dx =$

$-\frac{1}{6}\int \frac{1}{x-1}\,dx + \frac{1}{6}\int \frac{1}{x+1}\,dx + \frac{1}{12}\int \frac{1}{x-2}\,dx - \frac{1}{12}\int \frac{1}{x+2}\,dx =$

$-\frac{1}{6}\ln|x-1| + \frac{1}{6}\ln|x+1| + \frac{1}{12}\ln|x-2| - \frac{1}{12}\ln|x+2| + C$

8(b) $\displaystyle\int \frac{1}{x^4+1}\,dx$

$$x^4+1=(x^2-\sqrt{2}\,x+1)(x^2+\sqrt{2}\,x+1)$$

both are irreducible

$$\int \frac{1}{x^4+1}\,dx = \int \frac{1}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)}\,dx$$

$$\frac{1}{x^4+1} = \frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1}$$

$$1=(Ax+B)(x^2+\sqrt{2}\,x+1)+(Cx+D)(x^2-\sqrt{2}x+1)$$

$$1=Ax^3+\sqrt{2}\,Ax^2+Ax+Bx^2+\sqrt{2}Bx+B+$$
$$Cx^3-\sqrt{2}\,Cx^2+Cx+Dx^2-\sqrt{2}Dx+D$$

x^3: $0=A+C \longrightarrow C=-A$

x^2: $0=\sqrt{2}A+B+D-\sqrt{2}\,C$

x: $0=A+\sqrt{2}B+C-\sqrt{2}\,D$

constant: $1=B+D \longrightarrow D=1-B$

8 (b) continued

$0 = \sqrt{2}A + B + (1-B) - \sqrt{2}(-A)$

$0 = 2\sqrt{2}A + 1 \qquad 2\sqrt{2}A = -1 \quad A = \frac{-1}{2\sqrt{2}}$

$C = \frac{1}{2\sqrt{2}}$

$0 = A + \sqrt{2}B - A - \sqrt{2}(1-B)$

$0 = \sqrt{2}B - \sqrt{2} + \sqrt{2}B$

$\sqrt{2} = 2\sqrt{2}B \qquad B = \frac{1}{2} \qquad D = \frac{1}{2}$

$$\int \frac{\frac{-1}{2\sqrt{2}}x + \frac{1}{2}}{x^2 - \sqrt{2}x + 1}dx + \int \frac{\frac{1}{2\sqrt{2}}x + \frac{1}{2}}{x^2 + \sqrt{2}x + 1}dx =$$

$$-\frac{1}{2}\int \frac{\frac{1}{\sqrt{2}}x - 1}{x^2 - \sqrt{2}x + 1}dx + \frac{1}{2}\int \frac{\frac{1}{\sqrt{2}}x + 1}{x^2 + \sqrt{2}x + 1}dx =$$

$$-\frac{1}{2\sqrt{2}}\int \frac{x - \sqrt{2}}{x^2 - \sqrt{2}x + 1}dx + \frac{1}{2\sqrt{2}}\int \frac{x + \sqrt{2}}{x^2 + \sqrt{2}x + 1}dx$$

$$-\frac{1}{4\sqrt{2}}\int \frac{2x - 2\sqrt{2}}{x^2 - \sqrt{2}x + 1}dx + \frac{1}{4\sqrt{2}}\int \frac{2x + 2\sqrt{2}}{x^2 + \sqrt{2}x + 1}dx$$

8 b continued

$$-\frac{1}{4\sqrt{2}}\int\frac{2x-\sqrt{2}-\sqrt{2}}{x^2-\sqrt{2}x+1}dx+\frac{1}{4\sqrt{2}}\int\frac{2x+\sqrt{2}+\sqrt{2}}{x^2+\sqrt{2}x+1}dx$$

$$-\frac{1}{4\sqrt{2}}\int\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx+\frac{1}{4}\int\frac{1}{x^2-\sqrt{2}x+1}dx+$$

$$\frac{1}{4\sqrt{2}}\int\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx+\frac{1}{4}\int\frac{1}{x^2+\sqrt{2}x+1}dx$$

$u=x^2-\sqrt{2}x+1$ $\qquad$ $w=x^2+\sqrt{2}x+1$

$du=(2x-\sqrt{2})dx$ $\qquad$ $dw=(2x+\sqrt{2})dx$

$$-\frac{1}{4\sqrt{2}}\int\frac{du}{u}+\frac{1}{4\sqrt{2}}\int\frac{dw}{w}=-\frac{1}{4\sqrt{2}}\ln|u|+\frac{1}{4\sqrt{2}}\ln|w|$$

$$-\frac{1}{4\sqrt{2}}\ln|x^2-\sqrt{2}x+1|+\frac{1}{4\sqrt{2}}\ln|x^2+\sqrt{2}x+1|+$$

$$\frac{1}{4}\int\frac{1}{x^2-\sqrt{2}x+1}dx+\frac{1}{4}\int\frac{1}{x^2+\sqrt{2}x+1}dx$$

$x^2-\sqrt{2}x+1=(x-\frac{\sqrt{2}}{2})^2+1/2$ $\qquad$ $\int\frac{1}{a^2x^2+b^2}dx=\frac{1}{ab}\text{Arctan}\left(\frac{a}{b}x\right)$

$x^2+\sqrt{2}x+1=(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}$

8(b) continued

$$\frac{1}{4}\int \frac{1}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\,dx = \frac{\sqrt{2}}{4}\,\text{Arctan}\left(\sqrt{2}\left(x-\frac{\sqrt{2}}{2}\right)\right) =$$

$$\frac{\sqrt{2}}{4}\,\text{Arctan}(\sqrt{2}x-1)$$

$$\frac{1}{4}\int \frac{1}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\,dx = \frac{\sqrt{2}}{4}\,\text{Arctan}(\sqrt{2}x+1)$$

$$\int \frac{1}{x^4+1}\,dx = -\frac{1}{4\sqrt{2}}\ln|x^2-\sqrt{2}x+1| + \frac{1}{4\sqrt{2}}\ln|x^2+\sqrt{2}x+1| +$$

$$\frac{\sqrt{2}}{4}\,\text{Arctan}(\sqrt{2}x-1) + \frac{\sqrt{2}}{4}\,\text{Arctan}(\sqrt{2}x+1) =$$

$$\frac{2\,\text{Arctan}(\sqrt{2}x-1) + 2\,\text{Arctan}(\sqrt{2}x+1) + \ln(x^2+\sqrt{2}x+1) - \ln(x^2-\sqrt{2}x+1)}{4\sqrt{2}} + C$$

8ⓒ $\int \frac{1}{x^3-8}\,dx$ 2 is a root of x^3-8

$x-2$ is a factor of x^3-8

$$\begin{array}{r|l} & x^2+2x+4 \\ x-2 & x^3+0x^2+0x-8 \\ & -\ x^3-2x^2 \\ & \quad 2x^2+0x-8 \\ & \quad -\ 2x^2-4x \\ & \qquad 4x-8 \\ & \qquad -\ 4x-8 \\ & \qquad\quad 0 \end{array}$$

$x^3-8=(x-2)(x^2+2x+4)$

irreducible

$$\int \frac{1}{x^3-8}\,dx=\int \frac{1}{(x-2)(x^2+2x+4)}\,dx$$

$$\frac{1}{(x-2)(x^2+2x+4)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4}$$

$$1=A(x^2+2x+4)+(Bx+C)(x-2)$$

$\underline{x=2:}$ $1=12A$ $A=1/12$

$$1=\frac{1}{12}x^2+\frac{2}{12}x+\frac{4}{12}+Bx^2-2Bx+Cx-2C$$

x^2: $0=\frac{1}{12}+B$ $\underline{B=-1/12}$

x: $0=\frac{2}{12}-2\left(\frac{-1}{12}\right)+C$ $C=-4/12=-1/3$

8 (c) continued

$$\int \frac{1}{x^3-8}\,dx = \frac{1}{12}\int \frac{1}{x-2}\,dx + \int \frac{-\frac{1}{12}x-\frac{4}{12}}{x^2+2x+4}\,dx =$$

$$\frac{1}{12}\ln|x-2| - \frac{1}{12}\int \frac{x+4}{x^2+2x+4}\,dx =$$

$$\frac{1}{12}\ln|x-2| - \frac{1}{24}\int \frac{2x+8}{x^2+2x+4}\,dx =$$

$$\frac{1}{12}\ln|x-2| - \frac{1}{24}\int \frac{2x+2+6}{x^2+2x+4}\,dx =$$

$$\frac{1}{12}\ln|x-2| - \frac{1}{24}\int \frac{2x+2}{x^2+2x+4}\,dx - \frac{1}{4}\int \frac{1}{x^2+2x+4}\,dx$$

$$u = x^2+2x+4$$
$$du = (2x+2)\,dx$$

$$\frac{1}{12}\ln|x-2| - \frac{1}{24}\ln(x^2+2x+4) - \frac{1}{4}\int \frac{1}{(x+1)^2+3}\,dx =$$

$$\frac{1}{12}\ln|x-2| - \frac{1}{24}\ln(x^2+2x+4) - \frac{1}{4\sqrt{3}}\,\mathrm{Arctan}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

8(d) $\int \frac{2x+1}{x^2+4}\,dx = \int \frac{2x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx =$

$\ln(x^2+4) + \frac{1}{2}\operatorname{Arctan}\left(\frac{x}{2}\right) + C$

9(a) $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$ Trig sub

$x = \sin\theta$

$1-x^2 = \cos^2\theta$

$dx = \cos\theta\, d\theta$

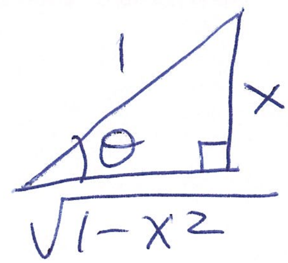
$\int \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} \cos\theta\, d\theta =$

$\int \sin^2\theta\, d\theta = \frac{1}{2}\int [1-\cos(2\theta)]\, d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right] =$

$\frac{1}{2}\left[\theta - \frac{1}{2} 2\sin\theta\cos\theta\right] = \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta$

$\theta = \operatorname{Arcsin}(x)$ $\sin\theta = x$

$\cos\theta = \sqrt{1-x^2}$



$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\operatorname{Arcsin}(x) -$

$\frac{1}{2}x\sqrt{1-x^2} + C$

9(b) $\int \frac{1}{\sqrt{6x-x^2-8}}\,dx$

$6x-x^2-8 = -x^2+6x-8 = (-1)(-1)[-x^2+6x-8] =$
$(-1)[x^2-6x+8] = (-1)[(x-3)^2-1] = 1-(x-3)^2$

$\int \frac{1}{\sqrt{6x-x^2-8}}\,dx = \int \frac{1}{\sqrt{1-(x-3)^2}}\,dx$ Trig sub

$(x-3) = \sin\theta \quad 1-(x-3)^2 = 1-\sin^2\theta = \cos^2\theta$

$dx = \cos\theta\,d\theta \quad \int \frac{\cos\theta}{\sqrt{\cos^2\theta d\theta}}\,d\theta = \int 1\,d\theta =$

$\theta \longrightarrow \text{Arcsin}(x-3)+C$

Can also use u-sub

$u = x-3 \quad du = dx \quad \int \frac{1}{\sqrt{1-(x-3)^2}}\,dx$ becomes

$\int \frac{1}{\sqrt{1-u^2}}\,du = \text{Arcsin}(u) \longrightarrow$

$\text{Arcsin}(x-3)+C$

9ⓒ $\int x \sin x \, dx$ by parts

$u = x \quad dv = \sin x \, dx \quad du = 1 dx \quad v = -\cos x$

$-x\cos x + \int \cos x \, dx = -x\cos x + \sin x + C$

9ⓓ $\int \frac{x}{\sqrt{1-x^2}} dx$ u-sub $u = 1 - x^2$

Also Trig sub $x = \sin\theta$

9ⓔ $\int \frac{x^2}{1-x^2} dx$ Long division → partial fractions

$1 - x^2 = (1-x)(1+x)$

Also trig sub $x = \sin\theta$

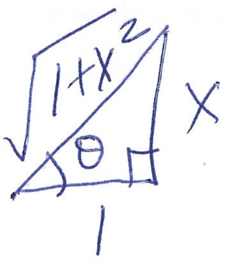
9ⓕ $\int (1+x^2)^{-3/2} dx = \int \frac{1}{(1+x^2)^{3/2}} dx$

Trig sub $x = \tan\theta \quad dx = \sec^2\theta \, d\theta$

$1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$

$\int \frac{1}{\sec^3\theta} \sec^2\theta \, d\theta = \int \frac{1}{\sec\theta} d\theta = \int \cos\theta \, d\theta =$

9(f) continued $\sin\theta + C$



Answer $\dfrac{x}{\sqrt{1+x^2}} + C$

9(g) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$ First u-sub

$u = x^2 \quad du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\displaystyle\frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$ Next trig sub

$u = \sin\theta$

9(h) $\displaystyle\int \frac{1}{1-x^2}\,dx = \int \frac{1}{(1-x)(1+x)}\,dx$

Partial fractions

Also by trig-sub $x = \sin\theta$

10 $\int_0^1 f(x)f'(x)\,dx$

u-sub $u=f(x)$ $du=f'(x)dx$

$x=0$ $u=f(0)=5$

$x=1$ $u=f(1)=6$

$$\int_5^6 u\,du=\frac{1}{2}u^2\Big|_5^6=\frac{1}{2}(36-25)=11/2$$

by PARts $\int_0^1 f(x)f'(x)dx$

$u=f(x)$ $dv=f'(x)dx$

$du=f'(x)dx$ $v=f(x)$

$$\int_0^1 f(x)f'(x)\,dx=f(x)f(x)\Big|_0^1-\int_0^1 f(x)f'(x)dx$$

$$2\int_0^1 f(x)f'(x)dx=[f(1)]^2-[f(0)]^2=36-25=11$$

$$\int_0^1 f(x)f'(x)dx=11/2$$

11 a) $\displaystyle\int_1^{\infty} \frac{\cos^2 x}{\sqrt{x^3}}\,dx$ $\qquad 0 \le \cos^2 x \le 1$

$\sqrt{x^3} = x^{3/2}$

$$0 \le \frac{\cos^2 x}{x^{3/2}} \le \frac{1}{x^{3/2}}$$

$$0 \le \int_1^{\infty} \frac{\cos^2 x}{x^{3/2}}\,dx \le \int_1^{\infty} \frac{1}{x^{3/2}}\,dx$$

converges p-integral with $p = \frac{3}{2} > 1$

Also converges by comparison test

---

11 b) $\displaystyle\int_3^{\infty} \frac{1}{x^2 \ln x}\,dx$ $\qquad x^2 \ln x > x^2$ for $x \ge 3$

$$\frac{1}{x^2 \ln x} < \frac{1}{x^2}$$

$$0 < \int_3^{\infty} \frac{1}{x^2 \ln x}\,dx < \int_3^{\infty} 1/x^2\,dx$$

converges p-integral $p = 2 > 1$

Also converges by comparison test.

11(c) $\int_1^{\infty} \frac{3+\sin x}{x^2}\,dx$ $\quad -1 \le \sin x \le 1$

$2 \le 3+\sin x \le 4$

$$2\int_1^{\infty} \frac{1}{x^2}\,dx < \int_1^{\infty} \frac{3+\sin x}{x^2}\,dx < 4\int_1^{\infty} \frac{1}{x^2}\,dx$$

Also converges by Comparison test.

converge p-integrals with $p=2>1$

11(d) $\quad -1 \le \sin x \le 1 \quad -2 \le 2\sin x \le 2$

$5+2\sin x \ge 5+(-2)=3$

$$\int_3^{\infty} \frac{5+2\sin x}{x-2}\,dx > 3\int_3^{\infty} \frac{1}{x-2}\,dx$$

Compare $\int_3^{\infty} \frac{1}{x-2}\,dx$ and $\int_3^{\infty} \frac{1}{x}\,dx$

11 (d) continued $\lim\limits_{x\to\infty} \dfrac{\frac{1}{x-2}}{\frac{1}{x}} = \lim\limits_{x\to\infty} \dfrac{x}{x-2} = 1$

by Limit Comparison Test, $\int_3^\infty \frac{1}{x-2}\,dx$

$\therefore$ diverges since $\int_3^\infty \frac{1}{x}\,dx$ diverges

Hence $\int_3^\infty \frac{5+2\sin x}{x-2}\,dx$ diverges since

$$\int_3^\infty \frac{5+2\sin x}{x-2}\,dx > 3\int_3^\infty \frac{1}{x-2}\,dx$$

---

11 (e) $\int_1^\infty \frac{\ln x}{x}\,dx$ $\quad u = \ln x \quad du = \frac{1}{x}\,dx$

$x=1 \quad u=0$

$x\to\infty \quad u=\ln x \to \infty$

$\int_0^\infty u\,du = \lim\limits_{b\to\infty} \int_0^b u\,du =$

$\lim\limits_{b\to\infty} \frac{1}{2}u^2 \Big|_0^b = \lim\limits_{b\to\infty} \frac{1}{2}b^2 \to \infty$

$\int_1^\infty \frac{\ln x}{x}\,dx$ diverges

11 (f) $\displaystyle\int_1^{\infty} \frac{1}{x\ln x}\,dx = \int_1^{2} \frac{1}{x\ln x}\,dx + \int_2^{\infty} \frac{1}{x\ln x}\,dx$

$$\int_1^{2} \frac{1}{x\ln x}\,dx = \lim_{a \to 1^+} \int_a^{2} \frac{1}{x\ln x}\,dx$$

$u = \ln x \quad du = \frac{1}{x}dx \quad \displaystyle\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du =$

$\ln \bar{u} \quad - \quad \ln(\ln x)$

$$\lim_{a \to 1^+} \ln(\ln x)\Big|_a^2 = \ln(\ln 2) - \lim_{a \to 1^+} \ln(\ln a) \to \alpha$$

$\displaystyle\int_1^{\infty} \frac{1}{x\ln x}\,dx$ diverges

---

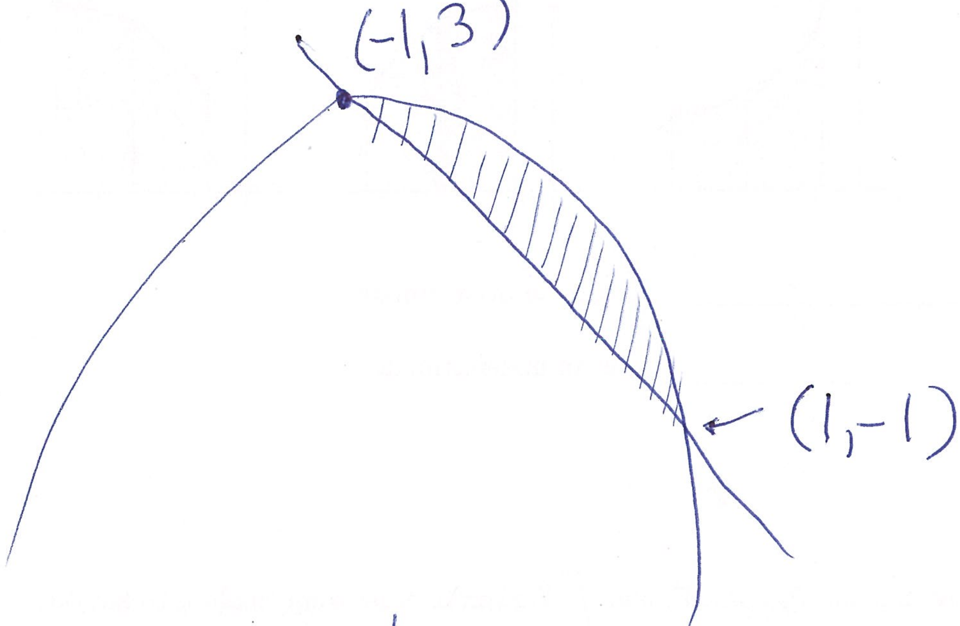
11 (g) $\displaystyle\int_{10}^{\infty} \frac{1}{x^2-9}\,dx$ compare with $\displaystyle\int_{10}^{\infty} \frac{1}{x^2}\,dx$

$$\lim_{x \to \infty} \frac{\frac{1}{x^2-9}}{\cancel{1/x^2}} = \lim_{x \to \infty} \frac{x^2}{x^2-9} = 1$$

Since $\displaystyle\int_{10}^{\infty} \frac{1}{x^2}\,dx$ converges, so does $\displaystyle\int_{10}^{\infty} \frac{1}{x^2-9}\,dx$ by LCT

(12) $-x^2-\cancel{2x}+2=1-\cancel{2x}$

$x^2=1 \quad x=\pm 1$

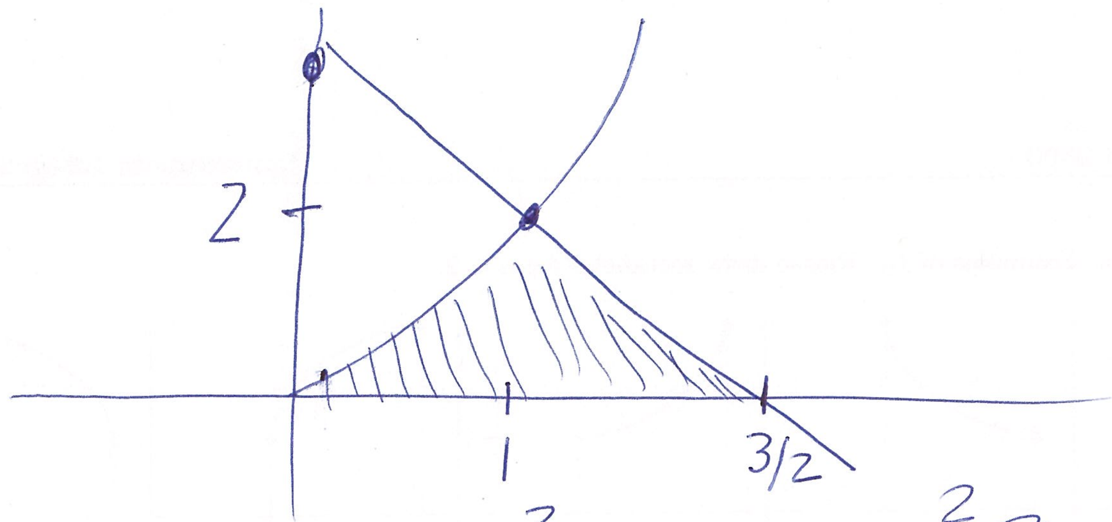


$$\text{Area}=\int_{-1}^{1}\left[(-x^2-2x+2)-(1-2x)\right]dx=$$

$$\int_{-1}^{1}(-x^2+1)\,dx=-\frac{1}{3}x^3+x\Big|_{-1}^{1}=$$

$$\left(-\frac{1}{3}+1\right)-\left(\frac{1}{3}-1\right)=2-\frac{2}{3}=4/3$$

13



$$2x^2 = -4x+6 \qquad x^2 = -2x+3 \qquad x^2+2x-3=0$$

$$(x+3)(x-1)=0 \qquad x=-3,\ 1 \quad \text{so } x=1$$

$$\text{Area} = \int_0^1 2x^2\,dx + \int_1^{3/2} (-4x+6)\,dx =$$

$$\frac{2}{3}x^3\Big|_0^1 + \left(-2x^2+6x\right)\Big|_1^{3/2} =$$

$$\frac{2}{3} + \left(-\frac{9}{2}+9\right) - (-2+6) = \frac{2}{3}+\frac{1}{2} = \frac{7}{6}$$

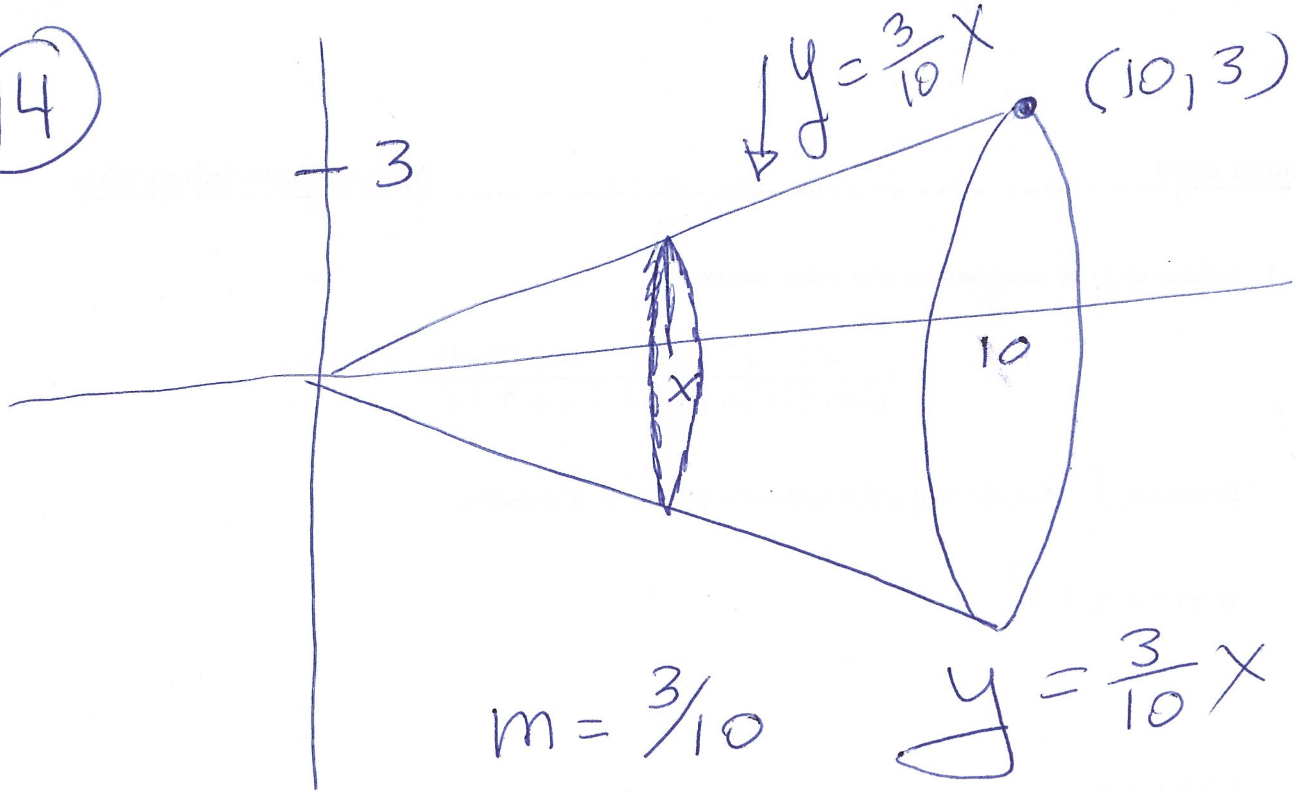
$$x = \frac{1}{\sqrt{2}}\sqrt{y} \qquad x = -\frac{1}{4}y+\frac{3}{2}$$

$$\int_0^2 \left(-\frac{1}{4}y+\frac{3}{2}-\frac{1}{\sqrt{2}}y^{1/2}\right)dy = -\frac{1}{8}y^2+\frac{3}{2}y-\frac{2}{3\sqrt{2}}y^{3/2}\Big|_0^2 =$$

$$-\frac{1}{2}+3-\frac{4\sqrt{2}}{3\sqrt{2}} = -\frac{1}{2}+3-\frac{4}{3} =$$

$$\frac{-3+18-8}{6} = 7/6$$

14



$$\text{Area} \cdot \Delta x = \text{volume of slice}(x)$$

$$\text{Area} = \pi r^2 = \pi\left(\tfrac{3}{10}x\right)^2 = \pi \tfrac{9}{100} x^2$$

$$\Delta V \approx \pi \tfrac{9}{100} x^2 \Delta x$$

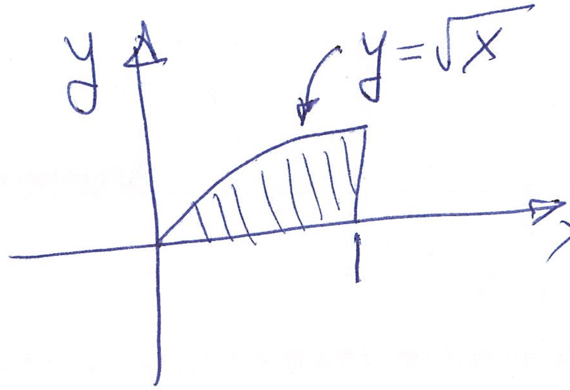
$$V = \int_0^{10} \frac{9\pi}{100} x^2 \, dx = \frac{9\pi}{100} \frac{1}{3} x^3 \Big|_0^{10} =$$

$$\frac{9\pi}{3} \cdot \frac{1000}{100} = 3\pi \cdot 10 = 30\pi$$

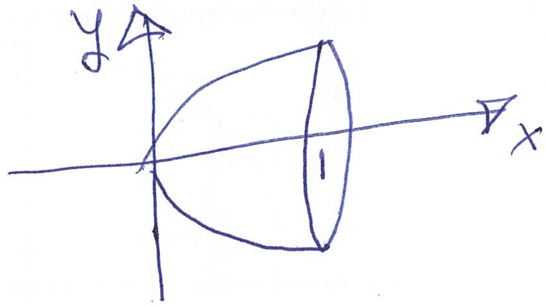
CHECK: Volume of cone with $r=3$ and $h=10$

is $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(10)9 = 30\pi$ ✓

15 (a)

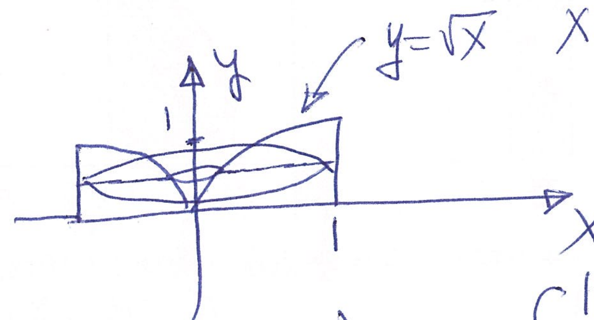


(b)

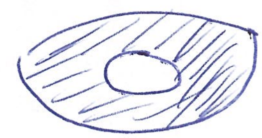


(c) $V = \pi\int_0^1 (\sqrt{x})^2 dx = \pi\int_0^1 x\,dx = \frac{1}{2}\pi x^2\Big|_0^1 = \pi/2$

(d)

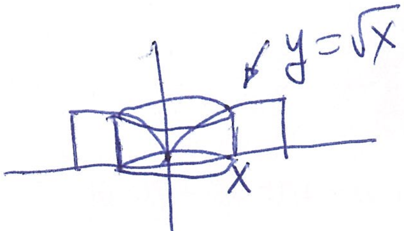


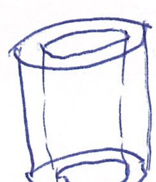
$x = y^2$



$V = \pi\int_0^1 [1^2 - (y^2)^2]\,\underline{dy}$

$\pi\left[y - \frac{1}{5}y^5\right]_0^1 = \frac{4}{5}\pi$ washer method



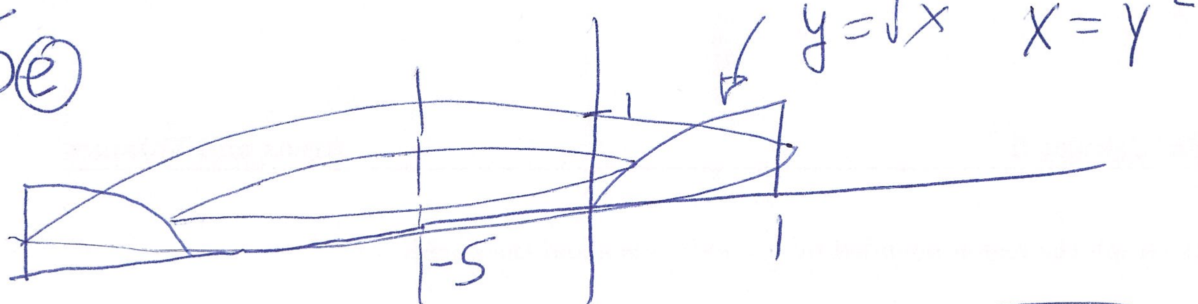


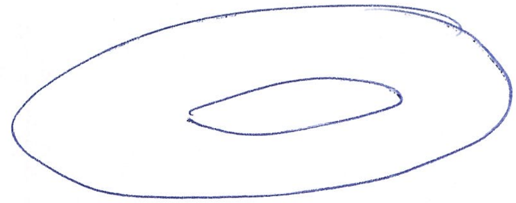
shell method

$V = 2\pi\int_0^1 x\sqrt{x}\,dx =$

$2\pi\int_0^1 x^{3/2}dx = \frac{4\pi}{5}x^{5/2}\Big|_0^1 = \frac{4}{5}\pi$

15(e)





Washer Method

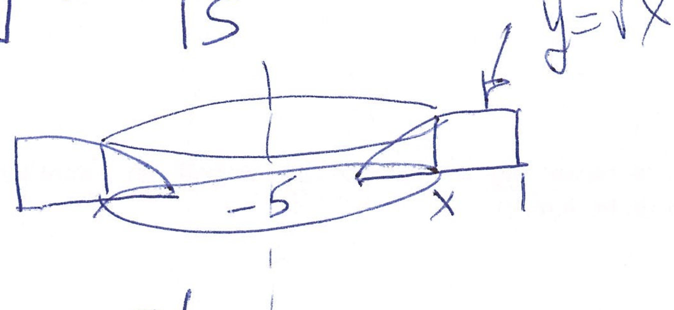
$$V=\pi\int_0^1\left[6^2-(5+y^2)^2\right]dy=$$

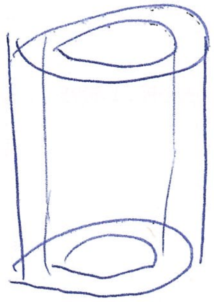
$$\pi\int_0^1\left[36-(25+10y^2+y^4)\right]dy=$$

$$\pi\int_0^1(11-10y^2-y^4)\,dy=\pi\left[11y-\frac{10}{3}y^3-\frac{1}{5}y^5\right]_0^1=$$

$$\pi\left[11-\frac{10}{3}-\frac{1}{5}\right]=\frac{112\pi}{15}$$

Shell Method



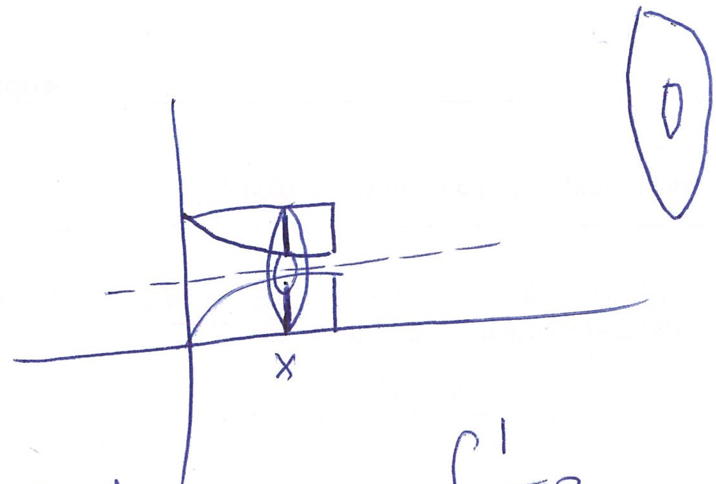


$$V=2\pi\int_0^1(5+x)\sqrt{x}\,dx=$$

$$2\pi\int_0^1\left(5x^{1/2}+x^{3/2}\right)dx=2\pi\left[\frac{10}{3}x^{3/2}+\frac{2}{5}x^{5/2}\right]_0^1=$$

$$2\pi\left[\frac{10}{3}+\frac{2}{5}\right]=2\pi\left[\frac{50+6}{15}\right]=\frac{112\pi}{15}$$

15(f)

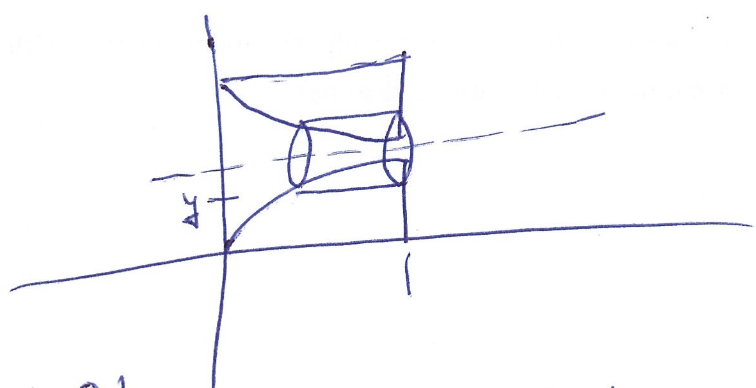


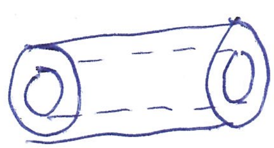
Washer method: $V = \pi \int_0^1 \left[1^2 - (1-\sqrt{x})^2\right] dx =$

$\pi \int_0^1 \left[1 - (1 - 2\sqrt{x} + x)\right] dx = \pi \int_0^1 (2\sqrt{x} - x)\, dx =$

$\pi \left[\frac{4}{3} x^{3/2} - \frac{1}{2} x^2\right]_0^1 = \pi\left[\frac{4}{3} - \frac{1}{2}\right] = \frac{5}{6}\pi$

Shell method



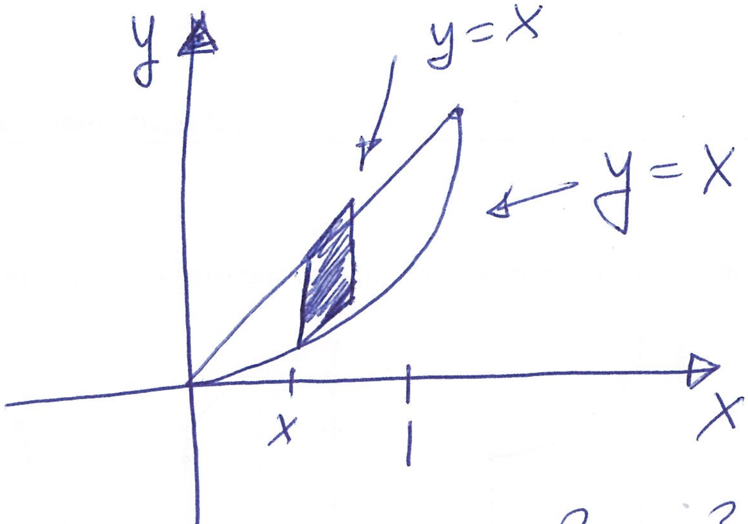


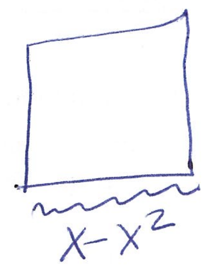
$$V = 2\pi \int_0^1 (1-y)(1-y^2)\, dy =$$

$2\pi \int_0^1 (1 - y - y^2 + y^3)\, dy = 2\pi \left[y - \frac{1}{2}y^2 - \frac{1}{3}y^3 + \frac{1}{4}y^4\right]_0^1$

$2\pi\left[1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4}\right] = 2\pi \, \frac{12-6-4+3}{12} = \frac{5\pi}{6}$

16





$$A(x) = (x - x^2)^2 = x^2 - 2x^3 + x^4$$

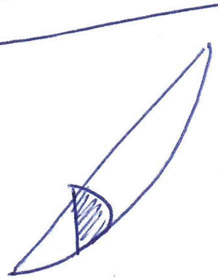
$$\Delta V \approx A(x)\Delta x$$

$$\text{Volume} = \int_0^1 (x^2 - 2x^3 + x^4)\,dx =$$

$$\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \Big|_0^1 =$$

$$\frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{10 - 15 + 6}{30} = \frac{1}{30}$$

17



radius $=$
$\frac{1}{2}(x - x^2)$

$$A(x) = \frac{1}{2}\pi r^2$$

$$A(x) = \frac{1}{2}\pi\left[\frac{1}{2}(x - x^2)\right]^2$$

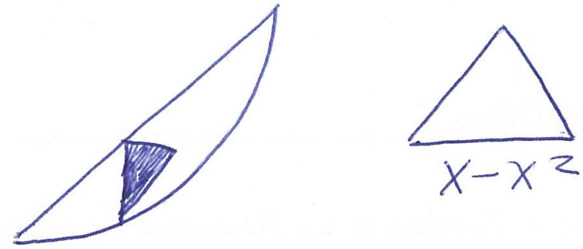
$$A(x) = \frac{1}{8}\pi (x - x^2)^2 =$$

$$\frac{1}{8}\pi (x^2 - 2x^3 + x^4)$$

$$V = \frac{1}{8}\pi \int_0^1 (x^2 - 2x^3 + x^4)\,dx =$$

$$\frac{1}{8}\pi \cdot \frac{1}{30} = \frac{\pi}{240}$$

(17) continued



$$A(x) = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}(x-x^2)^2 = \frac{\sqrt{3}}{4}(x^2-2x^3+x^4)$$

$$\text{Volume} = \frac{\sqrt{3}}{4}\int_0^1 (x^2-2x^3+x^4)\,dx =$$

$$\frac{\sqrt{3}}{4}\,\frac{1}{30} = \frac{\sqrt{3}}{120}$$

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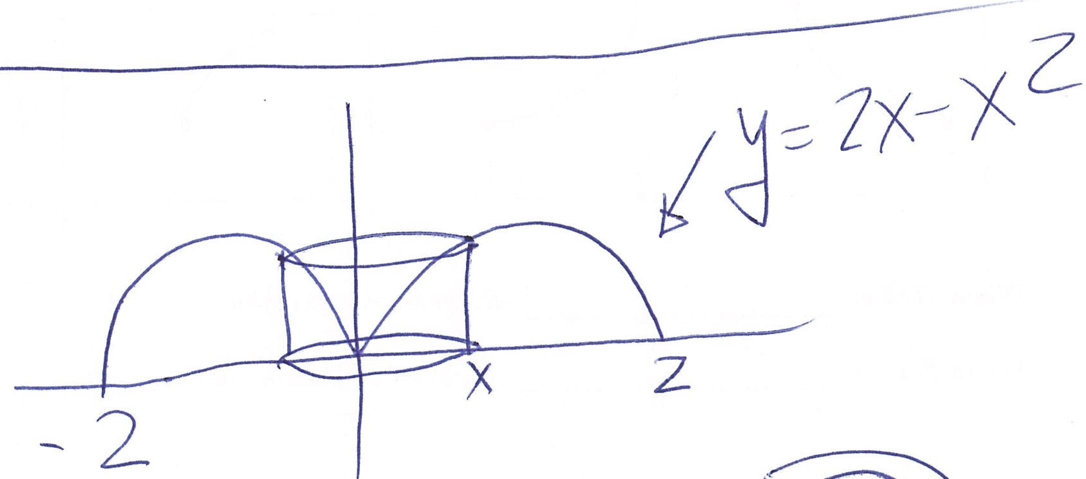
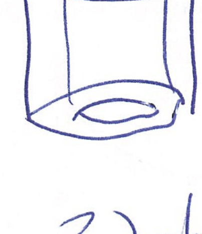
(18)



Shell method

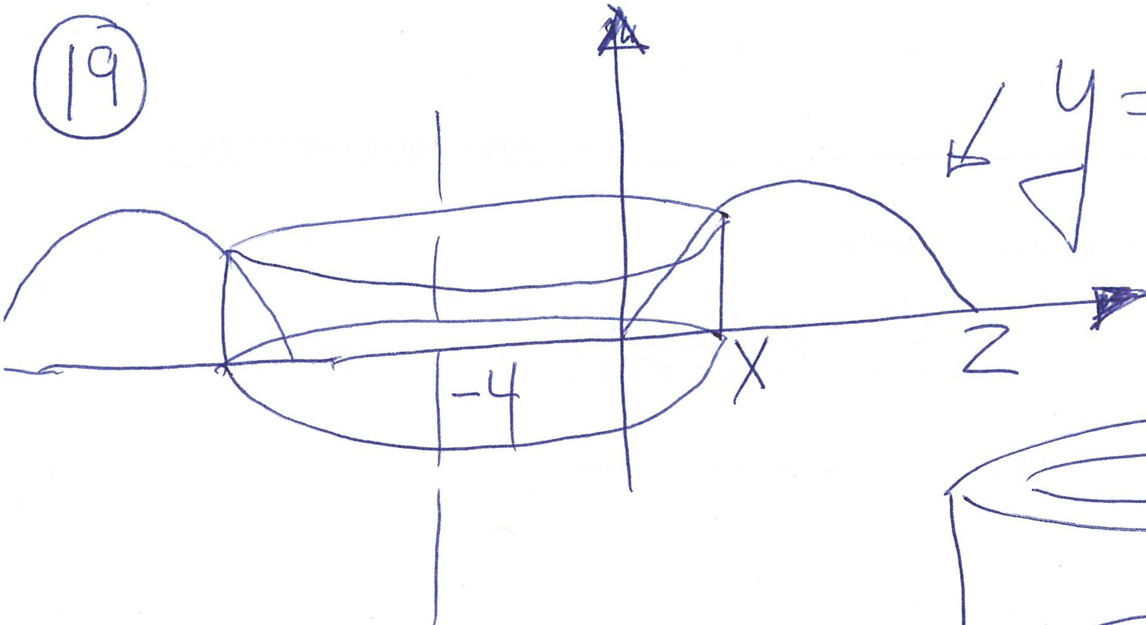


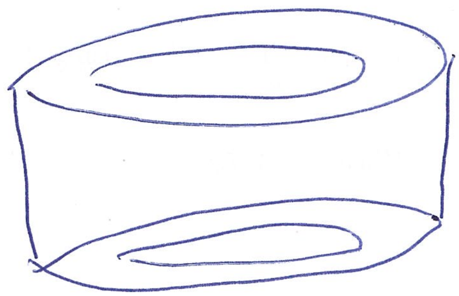
$$\text{Volume} = 2\pi\int_0^2 x(2x-x^2)\,dx =$$

$$2\pi\int_0^2 (2x^2-x^3)\,dx = 2\pi\left[\tfrac{2}{3}x^3-\tfrac{1}{4}x^4\right]_0^2 = 2\pi\left[\tfrac{16}{3}-\tfrac{12}{3}\right] = \frac{8\pi}{3}$$

19

$y = 2x^2 - x^3$





$$\text{Volume} = 2\pi \int_0^2 (4+x)(2x^2 - x^3)\,dx =$$

$$2\pi \int_0^2 (8x^2 - 4x^3 + 2x^3 - x^4)\,dx =$$

$$2\pi \int_0^2 (8x^2 - 2x^3 - x^4)\,dx =$$

$$2\pi \left[\frac{8}{3}x^3 - \frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^2 =$$

$$2\pi \left[\frac{64}{3} - 8 - \frac{32}{5}\right] = 2\pi \left[\frac{320 - 120 - 96}{15}\right] =$$

$$\frac{208}{15}\pi$$

20

$u = e^x \quad dv = f'(x)dx$

$du = e^x dx \quad v = f(x)$

$$\int_0^1 e^x f'(x)dx = e^x f(x)\Big|_0^1 - \int_0^1 e^x f(x)dx$$

but $f'(x) = f(x)$ so $\int_0^1 e^x f(x)dx =$

$$\int_0^1 e^x f'(x)dx$$

$$\int_0^1 e^x f'(x)dx = e^x f(x)\Big|_0^1 - \int_0^1 e^x f'(x)dx$$

$$2\int_0^1 e^x f'(x)dx = e^x f(x)\Big|_0^1$$

$$\int_0^1 e^x f'(x)dx = \frac{1}{2} e^x f(x)\Big|_0^1 =$$

$$\frac{1}{2}\left[e f(1) - f(0)\right] = \frac{1}{2}\left[e \cdot e - 1\right] = \frac{1}{2}\left[e^2 - 1\right]$$