Math 2300, Midterm 3

November 16, 2015

PRINT YOUR NAME: ______________________________

PRINT INSTRUCTOR'S NAME: ______________________________

Mark your section/instructor:

☐	Section 001	Albert Bronstein	9:00 - 9:50
☐	Section 002	Andrew Healy	10:00 - 10:50
☐	Section 003	Joshua Frinak	11:00 - 11:50
☐	Section 004	Kevin Berg	12:00 - 12:50
☐	Section 005	Jeffrey Shriner	2:00 - 2:50
☐	Section 006	Megan Ly	3:00 - 3:50
☐	Section 007	Albert Bronstein	8:00 - 8:50
☐	Section 008	Jonathan Lamar	1:00 - 1:50
☐	Section 009	Keli Parker	3:00 - 3:50
☐	Section 010	Steven Weinell	4:00 - 4:50
☐	Section 011	Benjamin Cooper	8:00 - 8:50
☐	Section 880	Jordan Watts	8:00 - 8:50

Question	Points	Score
1	8	
2	12	
3	12	
4	8	
5	12	
6	12	
7	12	
8	8	
9	6	
10	10	
Total:	100	

- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit. Use full mathematical or English sentences.
- You have 90 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like 100/7 or expressions like $\ln(3)/2$ as is.
- When done, give your exam to your instructor, who will mark your name off on a photo roster.
- We hope you show us your best work!

1. (8 points) Match the following functions with their corresponding Maclaurin series:

(a) $e^{x^2/2} = $ __VI__

(b) $\cos\left(\frac{x}{2}\right) = $ __II__

(c) $\frac{1}{(1-x)^2} = $ __III__

(d) $x\arctan(x) = $ __IV__

I. $\sum_{n=0}^{\infty} x^{2n}$

II. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(2n)!}$

III. $\sum_{n=1}^{\infty} n x^{n-1}$

IV. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+1}$

V. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$

VI. $\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$

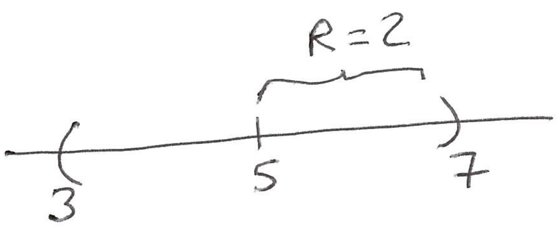
2. (12 points) (a) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{2^n n^2}$. Show all work in justifying your answer.

Ratio Test: $\lim_{n\to\infty} \dfrac{\dfrac{|x-5|^{n+1}}{2^{n+1}(n+1)^2}}{\dfrac{|x-5|^n}{2^n n^2}} = \lim_{n\to\infty} \dfrac{|x-5|^n |x-5|}{2\cdot 2^n (n^2+2n+1)} \dfrac{2^n n^2}{|x-5|^n}$

$\lim_{n\to\infty} |x-5| \dfrac{n^2}{2n^2+4n+2} = \frac{1}{2}|x-5| < 1$

$|x-5| < 2$ $\qquad -2 < x-5 < 2$

$3 < x < 7$



$R = 2$

(b) Find the interval of convergence. Show all work in justifying your answer.

Check End points $x=3$ and $x=7$

$x=3$: $\sum_{n=1}^{\infty} \dfrac{(3-5)^n}{2^n n^2} = \sum_{n=1}^{\infty} \dfrac{(-2)^n}{2^n n^2} = \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2}$

converges by AST

$x=7$ $\sum_{n=1}^{\infty} \dfrac{(7-5)^n}{2^n n^2} = \sum_{n=1}^{\infty} \dfrac{2^n}{2^n n^2} = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$

converges p-series with $p=2>1$.

Interval of convergence is $[3,7]$

3. (12 points) Find the solution of the differential equation

$$y(x+1) + y' = 0$$

that satisfies the initial condition $y(-2) = 1$. **Show all your work and write the solution on the line given below.**

$y(x+1) + y' = 0 \qquad y(x+1) + \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -y(x+1) \qquad \frac{dy}{y} = -(x+1)dx$

$\int \frac{dy}{y} = -\int (x+1)dx \qquad \ln|y| = -(\frac{1}{2}x^2 + x) + C$

$\ln 1 = -(\frac{1}{2}(-2)^2 + (-2)) + C$

$0 = C \qquad \ln|y| = -\frac{1}{2}x^2 - x$

$|y| = e^{-\frac{1}{2}x^2 - x}$

$y = \pm e^{-\frac{1}{2}x^2 - x}$ since $y(-2) = 1 > 0$

$y = e^{-\frac{1}{2}x^2 - x}$

Solution: $y =$ $e^{-\frac{1}{2}x^2 - x}$

4. (8 points) Given the following power series $\sum_{n=0}^{\infty} a_n(x-2)^n$ we know that at $x = 0$ the series converges and at $x = 8$ the series diverges. What do we know about the following values?

(a) At $x = 3$ the series $\sum_{n=0}^{\infty} a_n(x-2)^n$ is:

(i) convergent

(ii) divergent

(iii) We cannot determine its convergence/divergence with the information given.

(b) At $x = -4$ the series $\sum_{n=0}^{\infty} a_n(x-2)^n$ is:

(i) convergent

(ii) divergent

(iii) We cannot determine its convergence/divergence with the information given.

(c) At $x = 9$ the series $\sum_{n=0}^{\infty} a_n(x-2)^n$ is:

(i) convergent

(ii) divergent

(iii) We cannot determine its convergence/divergence with the information given.

(d) The following series $\sum_{n=0}^{\infty} a_n$ is:

(i) convergent

(ii) divergent

(iii) We cannot determine its convergence/divergence with the information given.

5. (12 points) (a) Write the definition for the n^{th} degree Taylor polynomial of $f(x)$ centered at $x = a$.

$$T_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 +$$

$$\frac{f'''(a)}{3!}(x-a)^3 + \frac{f''''(a)}{4!}(x-a)^4$$

(b) Find the second degree Taylor polynomial for $f(x) = \ln(\sec(x))$ centered at $\frac{\pi}{4}$.

$$f(\pi/4) = \ln(\sec(\pi/4)) = \ln(\sqrt{2}) = \tfrac{1}{2}\ln 2$$

$$f'(x) = \frac{\sec x \tan x}{\sec x} = \tan x \quad f'(\pi/4) = \tan \pi/4 = 1$$

$$f''(x) = \sec^2 x \quad f''(\pi/4) = \sec^2(\pi/4) = (\sqrt{2})^2 = 2$$

$$T_2(x) = \tfrac{1}{2}\ln 2 + (x - \pi/4) + \tfrac{2}{2}(x - \pi/4)^2$$

$$T_2(x) = \tfrac{1}{2}\ln 2 + (x - \pi/4) + (x - \pi/4)^2$$

6. (12 points) (a) Express the function $f(x) = \ln(1+x^3)$ as a power series centered about $x = 0$.

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x^3)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{3n}}{n}$$

(b) Express the definite integral as an infinite series.

$$\int_0^1 \ln(x^3+1)dx$$

$$\int_0^1 \ln(x^3+1)dx = \int_0^1 \left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{3n}}{n}\right)dx =$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \int_0^1 \frac{x^{3n}}{n} dx = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{3n+1}}{(3n+1)n} \Big|_0^1 =$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n(3n+1)}$$

7. (12 points) (a) Fill in the boxes to complete the statement of **Taylor's Inequality:**

If $\boxed{|f^{(n+1)}(x)|}$ $\le M$ on the interval between the center, a, and the point of approximation, x, then the remainder, $R_n(x)$, of the n^{th} degree Taylor polynomial, $T_n(x)$, satisfies the inequality:

$$|R_n(x)| \le \boxed{\frac{M}{(n+1)!}|x-a|^{n+1}}$$

(b) Use Taylor's Inequality to determine the number of terms of the Maclaurin series for e^x that should be used to estimate the number e with an error less than 0.6. Clearly justify your choice of M.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$f(x) = e^x$

$f^{(n+1)}(x) = e^x$

$x = 1$

So $M = e$ we can use $M = 3$

$|R_n(x)| \le \frac{3}{(n+1)!}|x|^{n+1}$ when $x = 1$

$|R_n(1)| \le \frac{3}{(n+1)!} \le 0.6 = \frac{6}{10} = \frac{3}{5}$

for $n = 2$ $\frac{3}{(2+1)!} = \frac{3}{3!} = \frac{3}{6} < \frac{3}{5}$

two terms of the Maclaurin series should be used.

8. (8 points) Each of the following slope fields represents one of the following differential equations. Match each slope field to the corresponding differential equation.

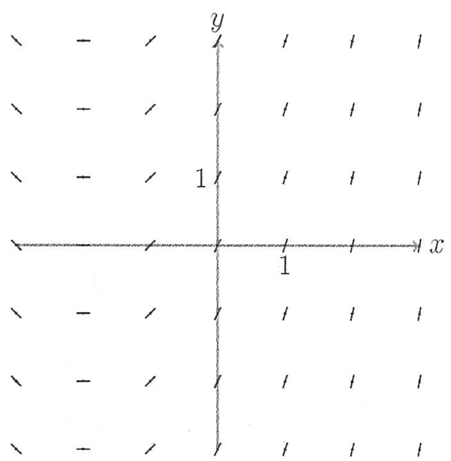
(a) $\dfrac{dy}{dx} = \dfrac{xy}{2}$ __II__

(b) $\dfrac{dy}{dx} = y - x - 2$ __III__

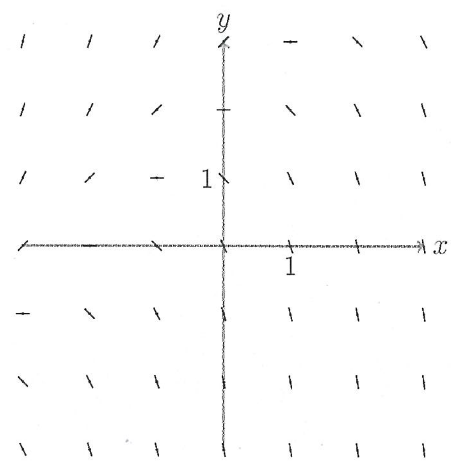
(c) $\dfrac{dy}{dx} = x + 2$ __I__

(d) $\dfrac{dy}{dx} = e^{x}$ __IV__

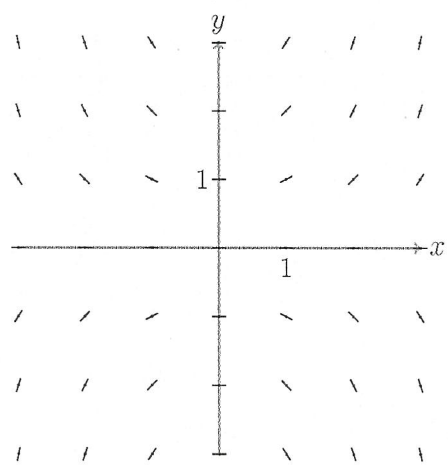
(I)



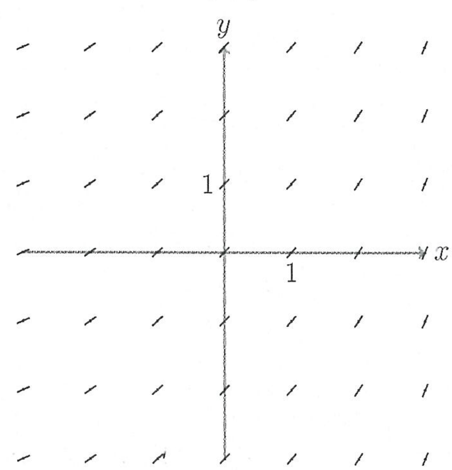
(III)



(II)



(IV)



9. (6 points) Find the sum of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

$$\operatorname{Arctan} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\frac{\pi}{4} = \operatorname{Arctan} 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

10. (10 points) Assume we approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{2}{n^2}$$

by using the first 3 terms. Give an upper bound for the error involved in this approximation by using the Remainder Estimate for the Integral Test.

$$S_3 = 2 + \frac{2}{4} + \frac{2}{9} = 2 + \frac{1}{2} + \frac{2}{9}$$

$$S = \sum_{n=1}^{\infty} 2/n^2 \qquad S - S_3 \leq \int_3^{\infty} \frac{2}{x^2}\,dx$$

$$\int_3^{\infty} \frac{2}{x^2}\,dx = \lim_{b \to \infty} \left. \frac{-2}{x} \right|_3^b = \lim_{b \to \infty} \frac{-2}{b} + \frac{2}{3} = \frac{2}{3}$$

$$\text{So } S - S_3 \leq \frac{2}{3}$$