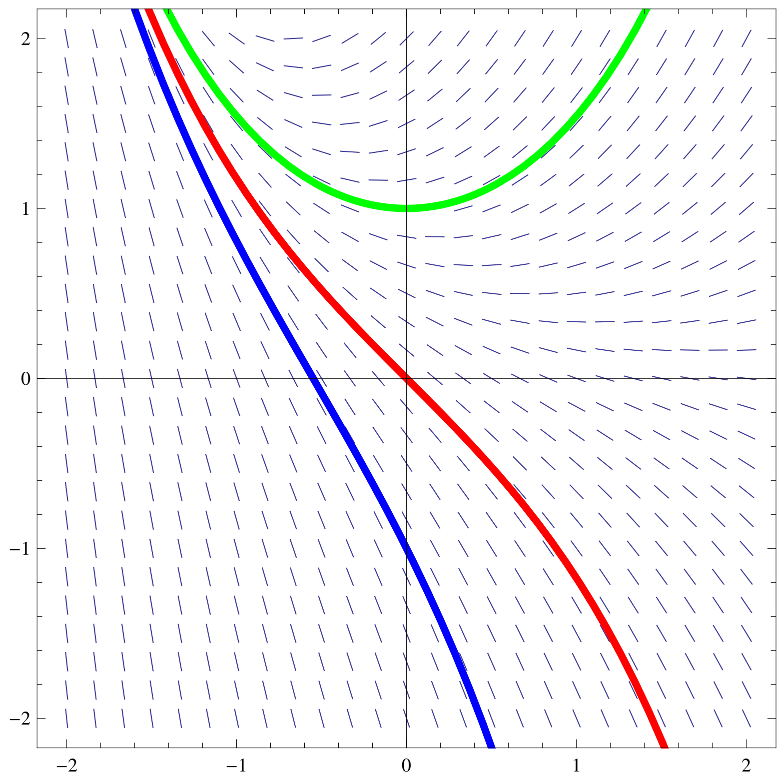
Math 2300 – Calculus II – University of Colorado

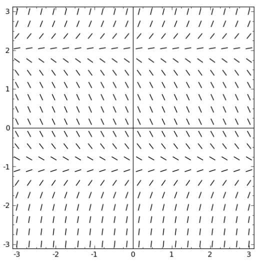
Final exam review problems

1. A slope field for the differential equation $y' = y - e^{-x}$ is shown. Sketch the graphs of the solutions that satisfy the given initial conditions. Make sure to label each sketched graph.

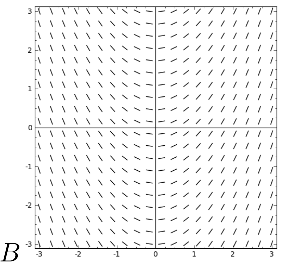
   (a) $y(0) = 0$ *RED* (b) $y(0) = 1$ *GREEN* (c) $y(0) = -1$ *BLUE*



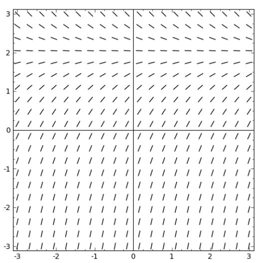
2. For each differential equation, find the corresponding slope field. (*Not all slope fields will be used.*)



A



B



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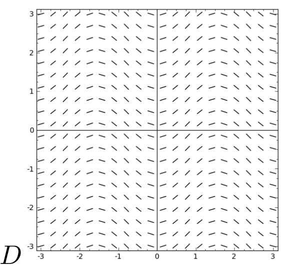

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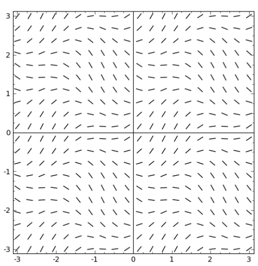
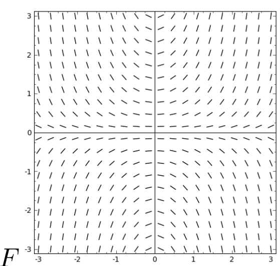


E



F

Equation	**Slope Field**
$\dfrac{dy}{dx} = x$	*B*
$y' = xy$	*F*
$y' = \sin(2x)$	*D*
$\dfrac{dy}{dx} = 2 - y$	*C*

3. For the previous problem, which slope fields have equilibrium solutions? Are they stable or unstable? A has an unstable equilibrium solution at $y = 2$ and a stable solution at $y = -1$. C has a stable equilibrium at $y = 2$ (which can be verified directly from its corresponding differential equation). F has an unstable equilibrium at $y = 0$ (which also can be determined from its corresponding differential equation.)

4. A bacteria culture contains 300 cells initially and grows at a rate proportional to its size. After half an hour the population has increased to 540 cells.

   (a) Write down the differential equation describing the situation.

   (b) Solve the differential equation

   (c) Use the remaining information in the problem to solve for the two constants, and write down the model for the number of bacteria at time t.

   (d) How long until there are 10000 bacteria?

5. Cobalt-60 has a half life of 5.24 years. We begin with a 50 mg sample.

   (a) Write down the applicable differential equation, given that radioactive substances decay at a rate proportional to the remaining mass.

   (b) Solve the differential equation, using the other information in the problem to determine the constants. Write down the model for the amount of Cobalt-60 remaining after t years.

   (c) How much remains after 20 years? How many years until 1 mg remains?

6. The rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. Write down the differential equation that models this statement, and solve it. Now a turkey at a room temperature of $70°F$ is put into a 350°F oven at 11:00 am. 1 hour later the meat

has a temperature of 100°F. I serve my cooked turkey at 160°F. If I need to remove the perfectly cooked turkey from the oven at 3:00, should I turn the oven temperature up or down? (Note: The temperatures given in this problem should not be considered official safety information. Consult the Food And Drug Agency's website for health recommendations for the cooking of poultry.)

7. How long will it take an investment to double if it is invested at 6% interested, compounded continuously? What about at 3%? Note that growth rate of an investment with continuously compounded interested are proportional to the current size of the investment.

8. The world population was estimated to be 190 million in the year 400 CE. Assuming a logistic growth model with a carrying capacity of 15 billion, and a growth rate of about 0.2% per year when the population was very small, write down the differential equation that models this situation.

9. The air in a room with volume 180 m^3 contains 0.15% carbon dioxide initially. Fresher air with only 0.05% carbon dioxide flows into the room at a rate of 2 m^3/min and the mixed air flows out at the same rate. Write down the initial value problem modeling this situation.

10. Graph the curve traced by each of the following parametric equations:
    (a) $x = 3\cos t, \qquad y = 2\sin t, \qquad 0 \le t \le 2\pi$
    (b) $x = 4\sin t, \qquad y = 4\cos t, \qquad 0 \le t \le \pi$
    (c) $x = -\cos t, \qquad y = 4\sin t, \qquad 0 \le t \le 2\pi$
    (d) $x = 2\cos 3t, \qquad y = -4\sin 3t, \qquad 0 \le t \le 2\pi$
    (e) $x = t\cos t, \qquad y = t\sin t, \qquad 0 \le t \le 6\pi$
    (f) $x = \cos t, \qquad y = \sin t,\ z = t, \qquad 0 \le t \le 4\pi$
    (g) $x = 4 + 5t, \qquad y = 6 - t,\ z = -2 + 3t, \qquad 0 \le t \le 1$

11. Create a parameterization for the described curves.
    (a) A circle starting at the point $(3, 0)$, traversing a circle centered at the origin, moving clockwise, traveling once around the circle.
    (b) Traverse the ellipse $x^2/9 + y^2/25 = 1$, start on the positive y-axis. Go in either direction, but state which direction your parameterization traverses.
    (c) A line segment starting at $(2, 3)$ and ending at $(-3, 5)$.

12. For the parameterized curve $x = t^3 - 3t, \qquad y = t^2 - 2t$:
    (a) Find the equation of the tangent line to the curve at $t = -2$.
    (b) Find d^2y/dx^2 at $t = -2$.
    (c) Find the speed at $t = -2$.
    (d) Is the tangent line to the curve ever vertical? When and where?
    (e) Is the tangent line to the curve ever horizontal? When and where?
    (f) Does the particle ever come to a stop?
    (g) Write down the integral giving arclength from $t = -2$ to $t = 1$.
    (h) What is the average speed from $t = -2$ to $t = 1$?

13. Find the area of the region bounded by the curve $r = \sqrt{\theta}$, for $0 \le \theta \le \pi$.

$$\int_0^\pi \frac{(\sqrt{\theta})^2}{2}\, d\theta = \frac{\pi^2}{4}$$

14. Find the area of the region that lies within the limaçon $r = 3 + 2\cos(\theta)$ and outside the circle $r = 4$.

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3} [(3 + 2\cos\theta)^2 - 4^2]\, d\theta = \frac{13\sqrt{3}}{2} - \frac{10\pi}{3}$$

15. Find the area of the region common to the circles $r = \cos(\theta)$ and $r = \sqrt{3}\sin(\theta)$.
(Hint: $\tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$.)

$$\frac{1}{2}\int_{0}^{\pi/6} (\sqrt{3}\sin\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} (\cos\theta)^2\, d\theta = \frac{1}{24}\left(5\pi - 6\sqrt{3}\right) \approx 0.221486$$

16. Find the exact length of the polar curve $r = e^{\theta}$, for $0 \leq \theta \leq \pi/2$.

$$\int_{0}^{\pi/2} \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta = \sqrt{2}\int_{0}^{\pi/2} e^{\theta}\, d\theta = \sqrt{2}(e^{\pi/2} - 1)$$