

MATH 2300 – review problems for Exam 3, part 1

1. Find the radius of convergence and interval of convergence for each of these power series:

(a) $\sum_{n=2}^{\infty} \frac{(x+5)^n}{2^n \ln n}$

Solution: Strategy: use the ratio test to determine that the radius of convergence is 2, so the endpoints are $x = -7$ and $x = -3$. At $x = -7$, we have the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$, use alternating series test (don't forget to show hypotheses are met) to show that this series converges. At $x = -3$ we have $\sum_{n=2}^{\infty} \frac{1}{\ln n}$, use term-size comparison test, comparing to $\sum_{n=2}^{\infty} \frac{1}{n}$ to show the series diverges. Interval of convergence is $[-7, -3)$.

(b) $\sum_{n=0}^{\infty} \frac{n(x-1)^n}{4^n}$

Solution: Strategy: use the ratio test to determine that the radius of convergence is 4, so the endpoints are $x = -3$ and $x = 5$. At $x = -3$ we have the series $\sum_{n=0}^{\infty} (-1)^n n$, which we can show diverges by the divergence test. At $x = 5$ we have the series $\sum_{n=0}^{\infty} n$, which we can also show diverges by the divergence test. The interval of convergence is $(-3, 5)$.

(c) $\sum_{n=0}^{\infty} n!(3x+1)^n$

Solution: Strategy: Use the ratio test to determine the radius of convergence.

$$\lim_{n \rightarrow \infty} \frac{(n+1)!(3x+1)^{n+1}}{n!(3x+1)^{n+1}} = \lim_{n \rightarrow \infty} \frac{n+1}{3x+1} = \infty$$

(provided $3x+1 \neq 0$). So the radius of convergence is 0. The only “endpoint” is when $3x+1 = 0$, or $x = -\frac{1}{3}$. At this point, the sum becomes $\sum_{n=0}^{\infty} 0$, which converges. So the interval of convergence is actually just a point, $x = -\frac{1}{3}$.

(d) $\sum_{n=0}^{\infty} \frac{(-2)^{n+1} x^n}{n^3 + 1}$

Solution: Strategy: use the ratio test to show the radius of convergence is $\frac{1}{2}$, so the endpoints are $x = -\frac{1}{2}$ and $x = \frac{1}{2}$. At $x = -\frac{1}{2}$ we have the series $-2 \sum_{n=0}^{\infty} \frac{1}{n^3 + 1}$. You can show this converges using term-size comparison, comparing to $\sum_{n=1}^{\infty} \frac{1}{n^3}$. At $x = \frac{1}{2}$ we have the series

$-2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n^3 + 1}$, which we can show converges absolutely by the term-size comparison test. The interval of convergence is $[-1/2, 1/2]$.

$$(e) \sum_{n=1}^{\infty} \frac{\ln nx^n}{n!}$$

Solution: Again, use the ratio test.

$$\lim_{n \rightarrow \infty} \frac{\ln(n+1) \cdot x^{n+1} \cdot n!}{\ln n \cdot x^n \cdot (n+1)!} = |x| \cdot \lim_{n \rightarrow \infty} \frac{1}{n+1} \cdot \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\ln n}.$$

Use L'Hopital's rule on the last limit, we get a limit of $|x| \cdot 0 \cdot 1 = 0$, regardless of the value of x . So the radius of convergence is infinite, and the interval of convergence is $(-\infty, \infty)$ (meaning that the series converges for all x).

2. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{(x+4)^n}{n^2}$$

Find the intervals of convergence of f and f' . For f : $[-5, -3]$. For f' : $[-5, -3]$.

- If $\sum b_n(x-2)^n$ converges at $x=0$ but diverges at $x=7$, what is the largest possible interval of convergence of this series? What's the smallest possible? Largest: $[-3, 7)$. Smallest: $[0, 4)$.
- The power series $\sum c_n(x-5)^n$ converges at $x=3$ and diverges at $x=11$. What are the possibilities for the radius of convergence? What can you say about the convergence of $\sum c_n$? Can you determine if the series converges at $x=6$? At $x=7$? At $x=8$? at $x=2$? At $x=-1$? At $x=-2$? At $x=12$? At $x=-3$? The radius of convergence must be between 2 and 6 (inclusive). When we substitute $x=6$, we get $\sum c_n$, which must converge since $x=6$ is inside the radius of convergence. The series converges at $x=6$. The series diverges at $x=-2$. We don't have enough information to determine convergence at $x=2$ or $x=8$. We also can't determine convergence at $x=-1$ or $x=7$, which possibly lie right at the edge of the interval of convergence.
- The series $\sum c_n(x+2)^n$ converges at $x=-4$ and diverges at $x=0$. What can you say about the radius of convergence of the power series? What can you say about the convergence of $\sum c_n$? What can you say about the convergence of the series $\sum c_n 2^n$? What can you say about the convergence/divergence of the series at $x=-1$? At $x=-3$? At $x=1$? At $x=-10$? This time the radius of convergence must be exactly 2, so the interval of convergence is $[-4, 0)$ When we substitute $x=-1$ we get $\sum c_n$, which must converge since $x=-1$ is within the interval of convergence. When we substitute $x=0$ we get $\sum c_n 2^n$, which we have been told diverges. The series converges at $x=-1$ and at $x=-3$ and diverges at $x=1$ and $x=10$.
- Say that $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. Find $f'(x)$ by differentiating termwise.
 $f'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$. Note that $f(x) = \sin x$, and $f'(x) = \cos x$.
- Use any method to find a power series representation of each of these functions, centered about $a=0$. Give the interval of convergence (Note: you should be able to give this interval based on your derivation of the series, not by using the ratio test.)

$$(a) \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\begin{aligned}
\text{(b)} \quad \frac{1}{1+x^2} &= \sum_{n=0}^{\infty} (-1)^n x^{2n} \\
\text{(c)} \quad \arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \\
\text{(d)} \quad xe^x - x &= \sum_{n=1}^{\infty} \frac{x^{n+1}}{n!} \\
\text{(e)} \quad \ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \\
\text{(f)} \quad x \ln(1+3x^2) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n x^{2n+1}}{n} \\
\text{(g)} \quad \frac{\sin(-2x^2)}{x} &= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^{2n+1} x^{4n+1}}{(2n+1)!} \\
\text{(h)} \quad \frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} nx^{n-1} \\
\text{(i)} \quad \int \frac{1}{1+x^5} dx &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+1}}{5n+1} + C
\end{aligned}$$

8. Determine the function or number represented by the following series:

$$\begin{aligned}
\text{(a)} \quad \sum_{n=1}^{\infty} nx^{n-1} &= \frac{1}{(1-x)^2} \\
\text{(b)} \quad \sum_{n=1}^{\infty} nx^n &= \frac{x}{(1-x)^2} \\
\text{(c)} \quad \sum_{n=0}^{\infty} \frac{x^{2n}}{5^{2n} n!} &= e^{x^2/25} \\
\text{(d)} \quad \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n+1}}{(2n+1)!} &= \frac{1}{2} \sin 2x \\
\text{(e)} \quad \sum_{n=1}^{\infty} \frac{x^{2n}}{n} &= -\ln(1-x^2) \\
\text{(f)} \quad \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!} &= \cos(3)
\end{aligned}$$

9. A car is moving with speed 20 m/s and acceleration 2 m/s² at a given instant. Using a second degree Taylor polynomial, estimate how far the car moves in the next second.

Solution: You should use the Taylor polynomial $P_2(x) = x^2 + 20x + C$ where C is some constant. Then the best estimation for how far the car moves in the next second is

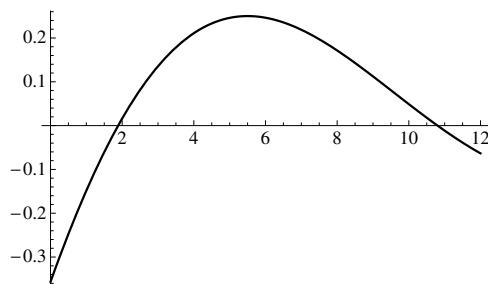
$$P_2(1) - P_2(0) = 21 + C - C = 21 \text{ meters}$$

10. Estimate $\int_0^1 \frac{\sin t}{t} dt$ using a 3rd degree Taylor Polynomial. What degree Taylor Polynomial should be used to get an estimate within 0.005 of the true value of the integral? (Hint: use the alternating series estimate). **Answer: $\frac{17}{18}$. The value is estimated by the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$, the n th term is less than 0.005 when $n = 2$, so we must add only two terms, $1 - \frac{1}{3 \cdot 3!}$**
11. Calculate the Taylor series of $\ln(1+x)$ by two methods. First calculate it “from scratch” by finding terms from the general form of Taylor series. Then calculate it again by starting with the Taylor series for $f(x) = \frac{1}{1-x}$ and manipulating it. Determine the interval of convergence each time.
12. Express the integral as an infinite series.

$$\int \frac{e^x - 1}{x} dx$$

$$\int \frac{e^x - 1}{x} dx = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} + C$$

13. Let $f(x) = \frac{1}{1-x}$.
- (a) Find an upper bound M for $|f^{(n+1)}(x)|$ on the interval $(-1/2, 1/2)$. **$2^{n+2} \cdot (n+1)!$**
- (b) Use this result to show that the Taylor series for $\frac{1}{1-x}$ converges to $\frac{1}{1-x}$ on the interval $(-1/2, 1/2)$. **By part (a) and by Taylor’s inequality, we have $|R_n(x)| \leq \frac{2^{n+2} \cdot (n+1)!}{(n+1)!} |x|^{n+1} = 2 \cdot (2|x|)^{n+1}$ on $(-1/2, 1/2)$. But the fact that $|x| < 1/2$ on this interval tells us that $\lim_{n \rightarrow \infty} |R_n(x)| \leq 2 \cdot \lim_{n \rightarrow \infty} (2|x|)^{n+1} = 0$ on this interval. But remember that $R_n(x) = f(x) - P_n(x)$, where $P_n(x)$ is the n th degree Taylor polynomial for $f(x)$. So $P_n(x) \rightarrow f(x)$ as $x \rightarrow \infty$, and we’re done.**
14. Consider the function $y = f(x)$ sketched below.



Suppose $f(x)$ has Taylor series

$$f(x) = a_0 + a_1(x-4) + a_2(x-4)^2 + a_3(x-4)^3 + \dots$$

about $x = 4$.

- (a) Is a_0 positive or negative? Please explain. **$a_0 > 0$, because the function is positive at $x = 4$.**
- (b) Is a_1 positive or negative? Please explain. **$a_1 > 0$, because the function is increasing at $x = 4$.**
- (c) Is a_2 positive or negative? Please explain. **$a_2 < 0$, because the function is concave down at $x = 4$.**

15. How many terms of the Taylor series for $\ln(1+x)$ centered at $x = 0$ do you need to estimate the value of $\ln(1.4)$ to three decimal places (that is, to within .0005)? We will use the error bound. The error bound corresponding to $P_n(0.4)$ is given by $\frac{M(0.4)^{n+1}}{(n+1)!}$, where M is the maximum of $|f^{n+1}(u)|$ on the interval $[0, 0.4]$. For $n \geq 1$, the derivatives of $f(x) = \ln(1+x)$ are given by the following formula:
 $f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ Clearly, $|f^{n+1}(u)| = \frac{n!}{(1+u)^{n+1}}$ is decreasing on the interval $[0, 0.4]$, so $M = \frac{n!}{(1+0)^{n+1}} = n!$ The error bound is then $\frac{n!(0.4)^{n+1}}{(n+1)!} = \frac{(0.4)^{n+1}}{n+1}$. The first n for which the error bound is smaller than 0.0005 is $n = 6$. (Note: sticking strictly to the method of the textbook, you would find the maximum of $|f^{n+1}(u)|$ on the interval $[-0.4, 0.4]$. In this method, substitute $x = -0.4$ to find the bound M .)
16. (a) Find the 4th degree Taylor Polynomial for $\cos x$ centered at $a = \pi/2$.
 $P_4(x) = -(x - \frac{\pi}{2}) + \frac{1}{6}(x - \frac{\pi}{2})^3$
 (b) Use it to estimate $\cos(89^\circ)$. $89^\circ = \frac{89\pi}{180}$, so $\cos(89^\circ) \approx -(-\frac{\pi}{180}) + \frac{1}{6}(-\frac{\pi}{180})^3 \approx 0.0174524064$
 (c) Use Taylor's inequality to determine what degree Taylor Polynomial should be used to guarantee the estimate to within .005. The $(n+1)$ st derivative of $\cos(x)$ is $\pm \sin x$ or $\pm \cos x$, so an upper bound for $f^{(n+1)}(x)$ is $M = 1$. $|E_n(\frac{89\pi}{180})| \leq \frac{1}{(n+1)!} |\frac{89\pi}{180} - \frac{90\pi}{180}|^{n+1}$. When $n = 1$ this quantity is $< .005$, so the first term gives an approximation to within 0.005.
17. (a) Find the 3rd degree Taylor Polynomial $P_3(x)$ for $f(x) = \sqrt{x}$ centered at $a = 1$ by differentiating and using the general form of Taylor Polynomials.
Solution:

$$P_3(x) = 1 + \frac{x-1}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{16}$$

 (b) Use the Taylor Polynomial in part (a) to estimate $\sqrt{1.1}$.
Solution:

$$\sqrt{1.1} \approx P_3(1.1) = 1.0488125$$

 (c) Use Taylor's inequality to determine how accurate is your estimate is guaranteed to be.
Solution: $|f^{(4)}(x)| = \frac{15}{16}x^{-7/2}$. This is a decreasing function on the interval $[1, 1.1]$, so its largest value occurs at $x = 1$. Thus I can use $f(1) = \frac{15}{16}$ for M . By Taylor's inequality, the absolute value of my error is bounded by $\frac{M}{4!}(x-a)^4 = \frac{15}{16 \cdot 24}(.1)^4 \approx 3.9 \times 10^{-6}$. (Note: sticking strictly to the method of the textbook, we find the maximum of $|f^{(4)}(x)| = \frac{15}{16}x^{-7/2}$ on the interval $[0.9, 1.1]$. In this method substitute $x = 0.9$ to find the bound M .)
18. Use Taylor's inequality to find a reasonable bound for the error in approximating the quantity $e^{0.60}$ with a third degree Taylor polynomial for e^x centered at $a = 0$. We are estimating e^x at $x = 0.6$. For $f(x) = e^x$, $n = 3$, $a = 0$, $x = 0.6$, Taylor's inequality gives the bound $\frac{Mx^4}{4!}$, where M is the maximum of $|f^{(4)}(x)| = |e^x|$ on the interval $(0, 0.6)$. Since $|f^{(4)}(x)| = e^x$ is an increasing function, its maximum on this interval occurs at the right-hand endpoint, so $M = e^{0.6}$. The bound is: $\frac{e^{0.6}(0.6)^4}{4!} < \frac{3^{0.6}(0.6)^4}{4!}$. (Note: sticking strictly to the method of the textbook, we would find the maximum of $|f^{(4)}(x)|$ on the interval $(-0.6, 0.6)$, and the same value of $M = e^{0.6}$ would work.)
19. Consider the error in using the approximation $\sin \theta \approx \theta - \theta^3/3!$ on the interval $[-1, 1]$. Where is the approximation an overestimate? Where is it an underestimate?

For $0 \leq \theta \leq 1$, the estimate is an underestimate (the alternating Taylor series for $\sin \theta$ is truncated after a negative term). For $-1 \leq \theta \leq 0$, the estimate is an overestimate (the alternating Taylor series is truncated after a positive term).

20. Write down from memory the Taylor Series centered around $a = 0$ for the functions e^x , $\sin x$, $\cos x$ and $\frac{1}{1-x}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ converges to } e^x \text{ on } (-\infty, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \text{ converges to } \sin x \text{ on } (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \text{ converges to } \cos x \text{ on } (-\infty, \infty)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \text{ converges to } \frac{1}{1-x} \text{ on } (-1, 1)$$

21. (a) Find the 4th degree Taylor Polynomial for $f(x) = \sqrt{x}$ centered at $a = 1$ by differentiating and using the general form of Taylor Polynomials.

$$P_4(x) = 1 + \frac{x-1}{2} - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$$

- (b) Use the previous answer to find the 4th degree T.P. for $f(x) = \sqrt{1-x}$ centered at $x = 0$.

$$\text{substitute } 1-x \text{ for } x, \text{ need 4th degree: } P_4(x) = 1 - \frac{x}{2} - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4$$

- (c) Use the previous answer to find the 3rd degree T.P. for $f(x) = \frac{1}{\sqrt{1-x}}$.

$$\text{Differentiate, multiply by } -2: P_3(x) = 1 + \frac{x}{2} + \frac{3}{8}x^2 + \frac{5}{16}x^3$$

- (d) Use the previous answer to find the 3rd degree T.P. for $f(x) = \frac{1}{\sqrt{1-x^2}}$.

$$\text{Substitute } x^2 \text{ for } x: P_3(x) = 1 + \frac{x^2}{2}, \text{ note that the } x^3 \text{ term is } 0.$$

- (e) Use the previous answer to find the 3rd degree T.P. for $f(x) = \arcsin x$.

$$\text{Integrate, substitute to verify that the constant term is } 0: P_3(x) = x + \frac{x^3}{6}$$