MATH 2300 – review problems for Exam 2

1. A metal plate of constant density ρ (in gm/cm^2) has a shape bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 1$.

   (a) Find the mass of the plate. Include units.

   (b) Find the center of mass of the plate. Include units.

2. Write down (but do not evaluate) a definite integral representing the arc length of the function $f(x) = \ln x$ between $x = 1$ and $x = 5$.

3. If $\int_1^5 f(x)\,dx = 7$ and $\int_5^4 f(x)\,dx = 5$, then find the average value of $f(x)$ on $[1, 4]$.

4. Find the average value of $f(x) = 3^{-x}$ on the interval $[1, 3]$.

5. **(Exercise 7 from Section 6.6 in Stewart's Calculus Concepts and Contexts)** Suppose that 2J of work is needed to stretch a spring from its natural length of 30 cm to a length of 42 cm.

   (a) How much work is needed to stretch the spring from 35 cm to 40 cm?
   (b) How far beyond its natural length will a force of 30 N keep the spring stretched?

6. **(Exercise 11 from Section 6.6 in Stewart's Calculus Concepts and Contexts)** Show how to approximate the required work by a Riemann sum. Then express the work as an integral and evaluate it: a heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft high.

   (a) How much work is done in pulling the rope to the top of the building?
   (b) How much work is done in pulling half the rope to the top of the building?

7. **(Exercise 15 from Section 6.6 in Stewart's Calculus Concepts and Contexts)** Show how to approximate the required work by a Riemann sum. Then express the work as an integral and evaluate it: a leaky 10-kg bucket is lifted from the ground to a height of 12m at a constant speed with a rope that weighs 0.8 kg/m. Initially the bucket contains 36 kg of water, but the water leads at a constant rate and finishes draining just as the bucket reaches the 12-m level. How much work is done?

8. Complete Exercise 19 from Section 6.6 on pg. 473 in Stewart's Calculus Concepts and Contexts

9. Find the limit of all the sequences in the sequence activity:
   http://math.colorado.edu/math2300/projects/SequencesPractice.pdf

10. Does $\{a_n\}$, where $a_n = \frac{1}{n}$, converge? If so, what does it converge to?

11. Decide whether each of the following sequences converges. If a series converges, what does it converge to? If not, why not?

    (a) The sequence whose n-th term is $a_n = 1 - \frac{1}{n}$.
    (b) The sequence whose n-th term is $b_n = \sqrt{n+1} - \sqrt{n}$.
    (c) The sequence whose n-th term is $c_n = \cos(\pi n)$.

(d) The sequence $\{d_n\}$, where $d_1 = 2$ and

$$d_n = 2d_{n-1} \quad \text{for } n > 1.$$

12. Find the sum of the series. For what values of the variable does the series converge to this sum?

(a) $1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \cdots$

(b) $y - y^2 + y^3 - y^4 + \cdots$

(c) $4 + z + \frac{z^2}{3} + \frac{z^3}{9} + \cdots$

13. For each of the following series, determine whether or not they converge. If they converge, determine what they converge to.

(a) $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}$

(b) $\sum_{n=2}^{\infty} 3\frac{4^{n+1}}{5^{n-4}}$.

(c) $\sum_{n=3}^{\infty} \frac{7(-\pi)^{2n-1}}{e^{3n+1}}$

(d) $\sum_{n=2}^{8} 4(.07)^{n+1}$

(e) $\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}$.

14. For each of the following series, determine if it converges absolutely, converges conditionally, or diverges. Completely justify your answers, including all details.

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n+5}$

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

(c) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^2}$

(d) $\sum_{n=1}^{\infty} \frac{2n^2(-3)^n}{n!}$

(e) $\sum_{n=1}^{\infty} \left(\frac{4 \cdot 2^n}{(-3)^{n+1}} + \frac{1}{2^n}\right)$

(f) $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n^3 + 2n}$

(g) $\sum_{n=1}^{\infty} \frac{n + 3n^5}{2n^7 + 3}$

(h) $\sum_{n=1}^{\infty} n^{\frac{1}{n}}$

(i) $\sum_{n=1}^{\infty} \arctan n$

(j) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

(k) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$

(l) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

15. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

    (a) Confirm using the Alternate Series Test that the series converges.

    (b) How many terms must be added to estimate the sum to within .0001?

    (c) Estimate the sum to within .0001.

16. How many terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ should be added to estimate the sum to within .01? No calculators.

17. Check whether the following series converge or diverge. In each case, give the answer for convergence, and name the test you would use. If you use a comparison test, name the series $\sum b_n$ you would compare to.

    (a) $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)}$

    (b) $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

    (c) $\sum_{n=1}^{\infty} \left(n+\frac{1}{n}\right)^n$

    (d) $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{5n^2}$

    (e) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ (hint: consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$)

    (f) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

    (g) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n+3)!}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

18. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$. Are the following statements true or false? Fully justify your answer.

   (a) The series converges by limit comparison with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.
   (b) The series converges by the ratio test.
   (c) The series converges by the integral test.

19. Consider the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$. Are the following statements true or false? Fully justify your answer.

   (a) The series converges by limit comparison with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.
   (b) The series converges by the ratio test.
   (c) The series converges by the integral test.
   (d) The series converges by the alternating series test.
   (e) The series converges absolutely.

20. Suppose the series $\sum a_n$ is absolutely convergent. Are the following true or false? Explain.

   (a) $\sum a_n$ is convergent.
   (b) The sequence a_n is convergent.
   (c) $\sum (-1)^n a_n$ is convergent.
   (d) The sequence a_n converges to 1.
   (e) $\sum a_n$ is conditionally convergent.
   (f) $\sum \frac{a_n}{n}$ converges.

21. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$$

You must justify your answer to receive credit.

22. A ball is dropped from a height of 10 feet and bounces. Assume that there is no air resistance. Each bounce is $\frac{3}{4}$ of the height of the bounce before.

(1) Find an expression for the height to which the ball rises after it hits the floor for the nth time.

(2) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the nth time.

(3) Using without proof the fact that a ball dropped from a height of h feet reaches the ground in $\sqrt{h}/4$ seconds: Will the ball bounce forever? If not, how long it will take for the ball to come to rest?

Want more practice? Here's some more!

23. In theory, drugs that decay exponentially always leave a residue in the body. However in practice, once the drug has been in the body for 5 half-lives, it is regarded as being eliminated. If a patient takes a tablet of the same drug every 5 half-lives forever, what is the upper limit to the amount of drug that can be in the body?

24. Let $\{f_n\}$ be the sequence defined recursively by $f_1 = 5$ and $f_n = f_{n-1} + 2n + 4$.

    (a) Check that the sequence g_n whose n-th term is $g_n = n^2 + 3n + 1$ satisfies this recurrence relation, and that $g_1 = 5$. (This tells us $g_n = f_n$ for all n.)

    (b) Use the result of part (a) to find f_{20} quickly.

25. Find the values of a for which the series converges/diverges:

    (a) $\sum_{n=1}^{\infty} \left(\frac{1}{2a}\right)^n$

    (b) $\sum_{n=1}^{\infty} \frac{1}{a}\left(\frac{1}{2}\right)^n$

    (c) $\sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^a$

    (d) $\sum_{n=1}^{\infty} (\ln a)^n$

    (e) $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^a}$

    (f) $\sum_{n=1}^{\infty} (1 + a^n)$

    (g) $\sum_{n=1}^{\infty} (1 + a)^n$

    (h) $\sum_{n=1}^{\infty} n^{\ln a}$

    (i) $\sum_{n=1}^{\infty} a^{\ln n}$

26. Using the table below, estimate the length of the curve given by $y = f(x)$ from $(3, 4)$ to $(6, 0.7)$.

x	3	3.5	4	4.5	5	5.5	6
$f(x)$	4	3.6	2.4	-1	-0.5	0	0.7
$f'(x)$	-0.8	-2.4	-6.8	1	1	1.4	-0.4

27. Determine if these sequences converge absolutely, converge conditionally or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

28. A steady wind blows a kite due east. The kite's height above ground from horizontal position $x = 0$ to $x = 80$ feet is given by

$$y = 150 - \frac{1}{40}(x-50)^2$$

Find the distance traveled by the kite. Just set up the integral - don't evaluate.

For each of the following statements, determine if it is true Always, Sometimes or Never.

29. If a sequence a_n converges, then the sequence $(-1)^n a_n$ also converges.

30. If a sequence $(-1)^n a_n$ converges to 0, then the sequence a_n also converges to 0.

31. The average value of a function is negative.

32. The geometric series $\displaystyle\sum_{n=1}^{\infty} 5r^{n-1}$ converges to $\displaystyle\frac{5}{1-r}$

33. The geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{c}{5^{n-1}}$ converges to $\displaystyle\frac{5c}{4}$

34. If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges.

35. If a series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, then $\displaystyle\lim_{n\to\infty} a_n = 0$

36. If the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$ is conditionally convergent, then the sequence a_n converges.

37. If the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

38. If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges conditionally.

39. If the series $\displaystyle\sum_{n=1}^{\infty} |a_n|$ diverges, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.