MATH 2300: Calculus II, Fall 2014
MIDTERM #3

Wednesday, November 12, 2014

YOUR NAME:

Circle Your CORRECT Section

◯ **001** M. Pelfrey .... (9am)
◯ **002** E. Angel .... (10am)
◯ **003** E. Angel .... (11am)
◯ **004** J. Harper .... (12pm)
◯ **005** B. Chhay .... (2pm)
◯ **006** S. Weinell .... (3pm)
◯ **007** C. Blakestad .... (8am)
◯ **008** P. Washabaugh .... (1pm)
◯ **009** J. Harper .... (3pm)
◯ **010** K. Parker .... (4pm)

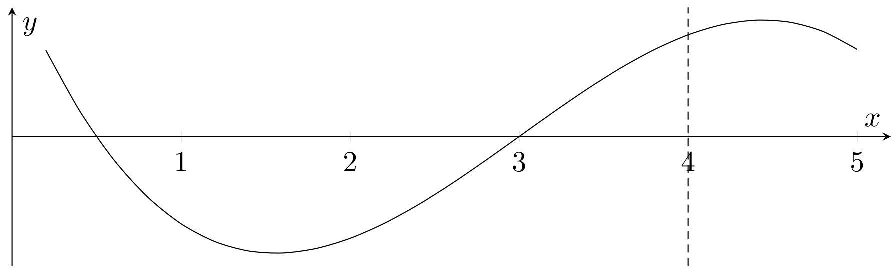
Important note: SHOW ALL WORK. BOX YOUR ANSWERS. Calculators are not allowed. No books, notes, etc. Throughout this exam, please provide exact answers where possible. That is: if the answer is 1/2, do not write 0.499 or something of that sort; if the answer is π, do not write 3.14159.

Problem	**Points**	**Score**
1	12	
2	8	
3	18	
4	10	
5	6	
6	12	
7	12	
8	12	
9	10	
TOTAL	100	

"On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work."

SIGNATURE:

1. (4 points each) Consider the function $y = f(x)$ sketched below.



Suppose $f(x)$ has Taylor series

$$f(x) = a_0 + a_1(x-4) + a_2(x-4)^2 + a_3(x-4)^3 + \cdots$$

about $x = 4$.

(a) Is a_0 positive or negative? Explain.

$a_0 = f(4)$ is positive because the graph is above the x-axis at $x = 4$.

(b) Is a_1 positive or negative? Explain.

$a_1 = f'(4)$ is positive because the graph has a horizontal tangent line at $x = 4$.

(c) Is a_2 positive or negative? Explain.

$a_2 = f''(4)/2$ is negative because the graph is concave down near $x = 4$.

2. (8 points) The power series $\sum_{n=0}^{\infty} c_n(x+7)^n$ converges when at $x = -1$ and diverges at $x = -16$. Determine the maximal and minimal possible interval of convergences.

The center of the power series is $x = -7$. Since the series converges at $x = -1$, the radius of convergence must be at least $|(-7) - (-1)| = 6$. The minimal possible interval of convergence is then

$$(-7-6, -7+6] = (-13, -1].$$

Since the series diverges at $x = -16$, the radius of convergence is at most $|(-7)-(-16)| = 9$. The maximal possible interval of convergence is then

$$(-7-9, -7+9] = (-16, 2].$$

3. (3 points each) Do the following series converge or diverge? **CIRCLE** your answer, no work necessary.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ CONVERGE **DIVERGE**

(Harmonic Series)

b) $\displaystyle\sum_{n=0}^{\infty} \frac{5}{4^n}$ **CONVERGE** DIVERGE

(Geometric Series)

c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\sqrt{n}}$ **CONVERGE** DIVERGE

(Alternating Series Test)

d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ CONVERGE **DIVERGE**

(Divergence Test or Ratio Test or Root Test)

e) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ **CONVERGE** DIVERGE

(Alternating Series Test)

f) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$ **CONVERGE** DIVERGE

(Ratio Test)

4. (10 points) Determine if the following series converges absolutely, conditionally, or diverges. **Justify your answer**.

$$\sum_{n=3}^{\infty} \frac{(-1)^n e^n}{(n-3)!}$$

Apply the ratio test: Let

$$L = \lim_n \left| \frac{(-1)^{n+1} e^{n+1}}{((n+1)-3)!} \cdot \frac{(n-3)!}{(-1)^n e^n} \right| = \lim_n \frac{e}{n-2} = 0$$

be the limit of the ratio of successive terms. Since $L = 0 < 1$, the series converges absolutely by the ratio test. Note we obtain the same limit with the absolute value of successive terms.

5. (6 points) Which of the following series could be the Taylor series of $f(t) = t\cos(2t^2)$ about $x = 0$?

(A) $t - \dfrac{2^2t^3}{2!} + \dfrac{2^4t^5}{4!} - \dfrac{2^6t^7}{6!} + \cdots$

(B) $t - \dfrac{2^2t^5}{2!} + \dfrac{2^4t^9}{4!} - \dfrac{2^6t^{13}}{6!} + \cdots$

(C) $2t^2 - \dfrac{2^3t^4}{3!} + \dfrac{2^5t^6}{5!} - \dfrac{2^7t^8}{7!} + \cdots$

(D) $2t^2 - \dfrac{2^3t^6}{3!} + \dfrac{2^5t^{10}}{5!} - \dfrac{2^7t^{14}}{7!} + \cdots$

(E) $t + 2t^2 + \dfrac{2^2t^3}{2!} + \dfrac{2^3t^4}{3!} + \cdots$

(F) $t + 2t^3 + \dfrac{2^2t^5}{2!} + \dfrac{2^3t^7}{3!} + \cdots$

Answer (B) is correct: Let $u = 2t^2$. Then

$$\begin{aligned}
t\cos(2t^2) = t\cos(u) &= t\left(1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots\right) \\
&= t\left(1 - \frac{(2t^2)^2}{2!} + \frac{(2t^2)^4}{4!} - \frac{(2t^2)^6}{6!} + \cdots\right) \\
&= t\left(1 - \frac{2^2t^4}{2!} + \frac{2^4t^8}{4!} - \frac{2^6t^{12}}{6!} + \cdots\right) \\
&= t - \frac{2^2t^5}{2!} + \frac{2^4t^9}{4!} - \frac{2^6t^{13}}{6!} + \cdots
\end{aligned}$$

6. (12 points) Use the integral test to determine if the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges or diverges.

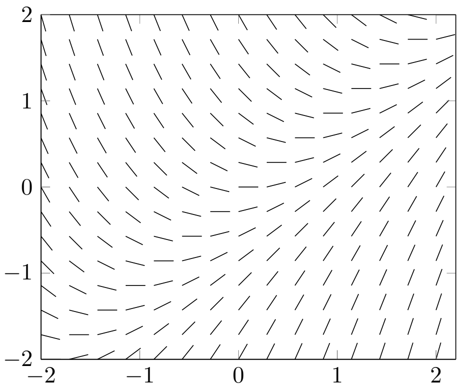
Note that $f(x) = \dfrac{1}{x(\ln x)^2}$ is positive and decreasing on $[2, \infty)$ as both x and $\ln x$ are increasing and positive on $[2, \infty)$, so that the hypothesis of the integral test is satisfied. We now integrate $f(x)$ using the u-substitution $u = \ln(x)$, $du = dx/x$.

$$\begin{aligned}
\int_2^{\infty} f(x)\,dx &= \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln(x))^2}\,dx = \lim_{b\to\infty} \int_{x=2}^b \frac{1}{u^2}\,du \\
&= \lim_{b\to\infty} \left.\frac{-1}{u}\right|_{x=2}^b = \lim_{b\to\infty} \left.\frac{-1}{\ln x}\right|_2^b = \lim_{b\to\infty}\left(\frac{-1}{\ln(b)} + \frac{1}{\ln(2)}\right) = \frac{1}{\ln(2)}.
\end{aligned}$$

Since $\displaystyle\int_2^{\infty} f(x)\,dx$ converges, the integral test implies $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

7. (2 points each) Match the slope fields pictured to the differential equations.

A



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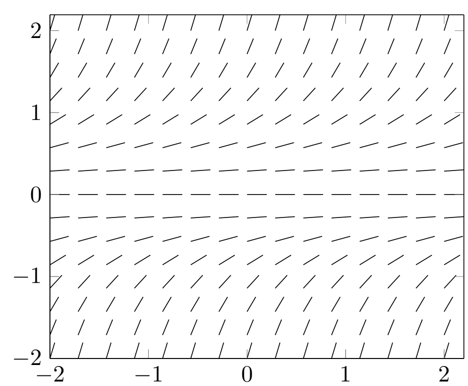

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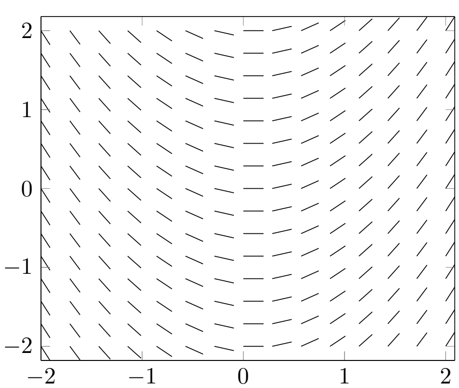
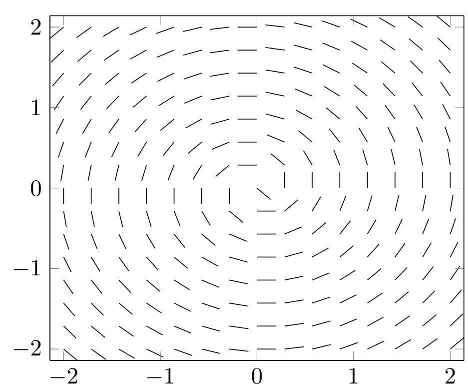
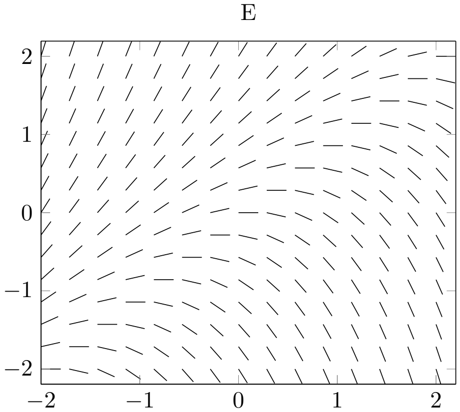


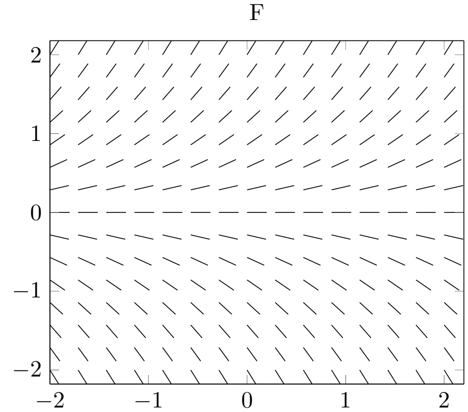
D



E



F



$\dfrac{dy}{dx} = -\dfrac{x}{y}$ $\underline{\text{D}}$ $\dfrac{dy}{dx} = x$ $\underline{\text{C}}$ $\dfrac{dy}{dx} = y^2$ $\underline{\text{B}}$

$\dfrac{dy}{dx} = y$ $\underline{\text{F}}$ $\dfrac{dy}{dx} = x - y$ $\underline{\text{A}}$ $\dfrac{dy}{dx} = y - x$ $\underline{\text{E}}$

8. (a) (5 points) Estimate $e^{0.1}$ using a third degree Taylor polynomial about 0.

The third degree Taylor polynomial of e^x about 0 is $P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$. So

$$P_3(0.1) = 1 + \frac{1}{10} + \frac{(1/10)^2}{2!} + \frac{(1/10)^3}{3!} = \frac{6631}{6000} = 1.1051\overline{6}.$$

(b) (7 points) Use the Lagrange error bound to find a bound on your approximation from part (a).

The Lagrange error bound states

$$|E_3(x)| \leq M\frac{|x|^4}{4!} \quad \text{for any} \quad M \geq |f^{(4)}(x)| \quad \text{on} \quad (0, 0.1)$$

We must choose an M which satisfies the inequality $M \geq e^x$ for $0 < x < 0.1$. Since e^x is increasing, the best (most precise) bound is $M = e^{0.1}$. On the other hand, this bound is not useful since $e^{0.1}$ is not an integer or rational, and is in fact precisely the number we are trying to approximate. $M = 3$ is a better bound for our purposes and is clearly greater than $e^{0.1} < (2.72)^{1/10}$. $M = 2$ is also greater than $e^{0.1}$. Using $M = 2$,

$$|E_3(0.1)| \leq 2\frac{0.1^4}{4!} = \frac{1}{120000} = 0.0000008\overline{3}.$$

9. (2 points each) Each of the series in part (a)-(e) converges to a value in the list below. Choose the correct value for each series from the list and write your choice in the blank.

(a)

$$1 + \frac{\pi}{1} + \frac{\pi^2}{2} + \frac{\pi^3}{3!} + \frac{\pi^4}{4!} + \cdots + \frac{\pi^n}{n!} + \cdots = e^{\pi}$$

(b)

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots = \ln(1+(1)) = \ln(2)$$

(c)

$$1 - \frac{\pi^2}{2} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \cdots + (-1)^n \frac{\pi^{2n}}{(2n)!} + \cdots = \cos(\pi) = -1$$

(d)

$$10 - \frac{10^3}{3!} + \frac{10^5}{5!} - \frac{10^7}{7!} + \cdots + (-1)^n \frac{10^{2n+1}}{(2n+1)!} + \cdots = \sin(10)$$

(e)

$$1 - \frac{\pi}{1} + \frac{\pi^2}{2} - \frac{\pi^3}{3!} + \frac{\pi^4}{4!} - \cdots + (-1)^n \frac{\pi^n}{n!} + \cdots = e^{-\pi}$$

- e^{π}
- 0
- 1
- $\ln(1+\pi)$
- -1
- $e^{-\pi}$
- $\ln(2)$
- $\sin(10)$
- $\cos(10)$
- e^{10}
- $\sin(1)$
- $\cos(1)$
- $\dfrac{1}{1-\pi}$
- $\pi/4$
- $\dfrac{1}{1-10}$
- $-\pi/4$