MATH 2300 – review problems for Sections 7.8-8.4

1. (a) Show that $\int_0^\infty e^{-x}dx$ is convergent.
   (b) Plot the functions $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$ on the same coordinate axes, for $x \geq 0$.
   (c) Is $\int_0^\infty e^{-x^2}dx$ convergent? (Hint: Use the fact that $x^2 \geq x$ for $x \geq 1$.) Justify your answers carefully.

2. Suppose that f is a continuous function, such that $f(x) \leq \dfrac{1}{x^2}$ for all $x \leq 100$. Does that imply $\int_1^\infty f(x)dx$ converges? Is it possible that $\int_1^\infty f(x)dx$ diverges? Justify your answer carefully.

3. Consider the plane region $\mathcal{R}$ enclosed between the curves $y = x^2$ and $y = x$, cut out of a steel plaque with variable density $\delta(x) = 1 + x$.
   (a) What is the area of the region?
   (b) What is the mass of the region?
   (c) What is the perimeter of the region? (You can leave your answer in simplified integral form.)

4. Consider the solid obtained by revolving the region $\mathcal{R}$ in the previous problem about the x-axis.
   (a) What is the volume of this solid?
   (b) What is its mass, if the density is constant in each vertical cross-section $\delta(x) = x$?
   (c) What are the volumes of the solids obtained by revolving the same region about the y-axis and about the horizontal line $y = 2$, respectively? Which is the largest volume of the three?

5. A sphere of radius $R = 2$ km is placed on the ground and filled with a gas that has density decreasing with the hight as $\delta(h) = 1/h^2$ kg/m^3, where h is the height from the ground. How high above the center of the sphere is the center of mass of the sphere, when it is completely filled with gas?

6. For the following curves given in polar form $r = f(\theta)$, plot the curve carefully and calculate the arc length and the area enclosed between the specified endpoints (set up the integral, simplify it and evaluate it, when possible):
   (a) $r = \theta$ for $0 \leq \theta = \pi/2$.
   (b) $r = 1$, $0 \leq \theta \leq \pi/3$
   (c) $r = 2\cos(\theta)$, $0 \leq \theta \leq \pi$

7. Review word problems in Section 9.2, such as # 27, #30 and #37 in the textbook.