MATH 2300: Calculus II, Fall 2014
MIDTERM #1

Wednesday, September 17, 2014

YOUR NAME:

Circle Your CORRECT Section

001 M. Pelfrey .... (9am)
002 E. Angel .... (10am)
003 E. Angel .... (11am)
004 J. Harper .... (12pm)
005 B. Chhay .... (2pm)
006 S. Weinell .... (3pm)
007 C. Blakestad .... (8am)
008 P. Washabaugh .... (1pm)
009 J. Harper .... (3pm)
010 K. Parker .... (4pm)

Important note: SHOW ALL WORK. BOX YOUR ANSWERS. Calculators are not allowed. No books, notes, etc. Throughout this exam, please provide exact answers where possible. That is: if the answer is 1/2, do not write 0.499 or something of that sort; if the answer is π, do not write 3.14159.

Problem	Points	Score
1	21	
2	7	
3	14	
4	12	
5	16	
6	20	
7	10	
TOTAL	100	

"On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work."

SIGNATURE:

NAME:	SECTION:

1. (7 points each) Compute the following indefinite integrals.

(a) $\int xe^x\,dx$

Use integration by parts with $u = x$ and $dv = e^x\,dx$ so that $du = dx$ and $v = e^x$. Thus

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

(b) $\int \tan^3(x)\sec(x)\,dx$

Rewrite the integral as follows:

$$\begin{aligned}\int \tan^3(x)\sec(x)\,dx &= \int \tan^2(x)(\tan x \sec x\,dx)\\ &= \int (\sec^2(x) - 1)(\tan x \sec x\,dx)\\ &= \int (u^2 - 1)(du) \qquad\qquad (u = \sec(x),\ du = \tan x \sec x)\\ &= \left(\frac{u^3}{3} - u\right) + C\\ &= \frac{\sec^3(x)}{3} - \sec(x) + C.\end{aligned}$$

(c) $\int (\cos x)(e^{\sin x})\,dx$

Let $u = \sin(x)$ so that $du = \cos(x)$. Then

$$\int (\cos x)(e^{\sin x})\,dx = \int e^u\,du = e^u + C = e^{\sin(x)} + C.$$

2. (7 points) Suppose that $\int_0^1 f(t)\,dt = 13$. Calculate $\int_{0.1}^{0.2} f(10t-1)\,dt$. Choose the best answer below.

A. 13 B. 1.3 C. 12 D. 1.2 E. 129

Let $u = 10t - 1$ so that $du = 10\,dt$. Further, if $t = 0.1$ then $u = 0$ and if $t = 0.2$ then $u = 1$. So

$$\begin{aligned}\int_{0.1}^{0.2} f(10t-1)\,dt &= \int_{u=0}^{1} f(u)\left(\frac{du}{10}\right)\\ &= \frac{1}{10}\int_0^1 f(u)\,du\\ &= \frac{13}{10} = 1.3\end{aligned}$$

Therefore B. is the best answer.

3. (7 points each) Compute the following indefinite integrals.

(a) $\int e^x \sin(4x)\,dx$

We will apply integration by parts twice. At first, let $u_1 = e^x$ and $dv_1 = \sin(4x)\,dx$ so that $du_1 = e^x\,dx$ and $v_1 = -\frac{1}{4}\cos(4x)$

$$\begin{aligned}\int e^x \sin(4x)\,dx &= e^x\left(-\frac{1}{4}\cos(4x)\right) - \int\left(-\frac{1}{4}\cos(4x)\right)e^x\,dx \\ &= -\frac{1}{4}e^x\cos(4x) + \frac{1}{4}\int e^x\cos(4x)\,dx\end{aligned}$$

Now let $u_2 = e^x$ and $dv_2 = \cos(4x)\,dx$ so that $du_2 = e^x\,dx$ and $v_2 = \frac{1}{4}\sin(4x)$ yielding

$$\begin{aligned} &= -\frac{1}{4}e^x\cos(4x) + \frac{1}{4}\left[e^x\left(\frac{1}{4}\sin(4x)\right) - \int\left(\frac{1}{4}\sin(4x)\right)e^x\,dx\right] + C \\ &= -\frac{1}{4}e^x\cos(4x) + \frac{1}{16}e^x\sin(4x) - \frac{1}{16}\int e^x\sin(4x)\,dx + C\end{aligned}$$

Finally, we can add $\frac{1}{16}\int e^x \sin(4x)\,dx$ to both sides of the equality to obtain

$$\begin{aligned}\frac{17}{16}\int e^x\sin(4x)\,dx &= -\frac{1}{4}e^x\cos(4x) + \frac{1}{16}e^x\sin(4x) + C \\ \int e^x\sin(4x)\,dx &= -\frac{4}{17}e^x\cos(4x) + \frac{1}{17}e^x\sin(4x) + C\end{aligned}$$

(b) $\int x^2(x+5)^{25}\,dx$

Let $u = x+5$ so that $du = dx$ and $x = u-5$. Then

$$\begin{aligned}\int x^2(x+5)^{25}\,dx &= \int (u-5)^2u^{25}\,du \\ &= \int\left(u^{27} - 10u^{26} + 25u^{25}\right)\,du \\ &= \frac{u^{28}}{28} - \frac{10u^{27}}{27} + \frac{25u^{26}}{26} + C \\ &= \frac{(x+5)^{28}}{28} - \frac{10(x+5)^{27}}{27} + \frac{25(x+5)^{26}}{26} + C.\end{aligned}$$

4. (6 points each) Parts (a) and (b) refer to the following functions:

I. $f(x) = -x^3 + 3$ II. $f(x) = \sin x + 1$ III. $f(x) = e^x$

(a) For which of the functions is TRAP(8) an overestimate for the integral of the function on the interval $[0, 1]$? Choose the best answer.
A) I
B) II
C) III
D) I and II
E) II and III
F) I and III
G) I, II, and III

Since, on $[0, 1]$, I. is concave down, II. is concave down, and III. is concave up, TRAP(8) is an overestimate for only III. Thus C is the best answer.

(b) For which of the functions is MID(8) an underestimate for the integral of the function on the interval $[-1, 0]$? Choose the best answer.
A) I
B) II
C) III
D) I and II
E) II and III
F) I and III
G) I, II, and III

Since, on $[-1, 0]$, I. is concave up, II. is concave up, and III. is concave up, MID(8) is an underestimate for all three functions. Thus G is the best answer.

5. (8 points each) Do the following integrals converge or diverge? Justify your answer.

(a) $\displaystyle\int_{25}^{\infty} \frac{1}{\sqrt{z}-4}\,dz$

Consider the function $\frac{1}{\sqrt{z}}$. On $[25, \infty)$, $\sqrt{z} - 4 < \sqrt{z}$ so that

$$\frac{1}{\sqrt{z}-4} > \frac{1}{\sqrt{z}}.$$

Since integrals of the form $\displaystyle\int_a^{\infty} \frac{1}{z^p}\,dz$ diverge for $a > 0$ and $p \leq 1$,

$$\int_{25}^{\infty} \frac{1}{\sqrt{z}}\,dz \quad \text{diverges.}$$

Hence, by the comparison test, $\displaystyle\int_{25}^{\infty} \frac{1}{\sqrt{z}-4}\,dz$ diverges.

(b) $\displaystyle\int_{2}^{\infty} \frac{d\theta}{\sqrt{\theta^3+1}}$

Consider the function $\frac{1}{\sqrt{\theta^3}}$. On $[2, \infty)$, $\sqrt{\theta^3+1} > \sqrt{\theta^3}$ so that

$$\frac{1}{\sqrt{\theta^3+1}} < \frac{1}{\sqrt{\theta^3}}.$$

Since integrals of the form $\displaystyle\int_a^{\infty} \frac{1}{\theta^p}\,d\theta$ converge for $a > 0$ and $p > 1$,

$$\int_{2}^{\infty} \frac{1}{\sqrt{\theta^3}}\,d\theta \quad \text{converges.}$$

Hence, by the comparison test, $\displaystyle\int_{2}^{\infty} \frac{d\theta}{\sqrt{\theta^3+1}}$ converges.

6. (10 points each) Find the following integrals.

(a) $\displaystyle\int_0^1 3\ln x\,dx$

As $\ln x$ diverges as $x \to 0^+$,

$$\int_0^1 3\ln x\,dx = \lim_{a\to 0^+}\int_a^1 3\ln x\,dx$$

and integration by parts shows, with $u = \ln x$ and $dv = dx$,

$$\int 3\ln x\,dx = 3\left(x\ln x - \int x\frac{1}{x}\,dx\right) + C = 3(x\ln x - x) + C$$

we can use the Fundamental Theorem of Calculus to write

$$\begin{aligned}\int_0^1 3\ln x\,dx &= \lim_{a\to 0^+}\int_a^1 3\ln x\,dx\\ &= \lim_{a\to 0^+}\left[3(x\ln x - x)\right]_a^1\\ &= \lim_{a\to 0^+} 3\left[((1)\ln(1) - 1) - (a\ln a - a)\right]\end{aligned}$$

as $\lim_{a\to 0^+}(a\ln a)$ is an indeterminate form, we rewrite and use L'Hôpital's rule

$$\begin{aligned}&= \lim_{a\to 0^+} 3\left(-1 + a - \frac{\ln a}{1/a}\right)\\ &= 3(-1) - 3\lim_{a\to 0^+}\left(\frac{1/a}{-1/a^2}\right)\\ &= -3 - 3\lim_{a\to 0^+}(-a) = -3.\end{aligned}$$

(b) $\displaystyle\int_{-2}^1 \frac{1}{\sqrt{5-4x-x^2}}\,dx$

The corresponding indefinite integral is found by completing the square and a sin sub:

$$\begin{aligned}\int \frac{1}{\sqrt{5-4x-x^2}}\,dx &= \int \frac{1}{\sqrt{-(x^2+4x-5)}}\,dx\\ &= \int \frac{1}{\sqrt{-(x^2+4x+4)+9}}\,dx\\ &= \int \frac{1}{\sqrt{3^2-(x+2)^2)}}\,dx\\ &= \int \frac{3\cos\theta\,d\theta}{\sqrt{3^2-(3\sin\theta)^2)}} && (x+2 = 3\sin\theta)\\ &= \int d\theta = \theta + C = \arcsin\left(\frac{x+2}{3}\right) + C.\end{aligned}$$

Hence, as the denominator of the integrand is zero when $x = 1$,

$$\begin{aligned}\int_{-2}^1 \frac{1}{\sqrt{5-4x-x^2}}\,dx &= \lim_{a\to 1^-}\int_{-2}^a \frac{1}{\sqrt{5-4x-x^2}}\,dx = \lim_{a\to 1^-}\left[\arcsin\left(\frac{x+2}{3}\right)\right]_{-2}^a =\\ &= \lim_{a\to 1^-}\arcsin\left(\frac{a+2}{3}\right) - \arcsin\left(\frac{(-2)+2}{3}\right) = \arcsin(1) - \arcsin(0) = \frac{\pi}{2}.\end{aligned}$$

7. (10 points) Compute the indefinite integral $\displaystyle\int \frac{3x^2 - 16x + 6}{(x+2)(x-3)^2}\, dx.$

The partial fractions decomposition for the integrand is

$$\frac{3x^2 - 16x + 6}{(x+2)(x-3)^2} = \frac{A}{x+2} + \frac{B_1}{x-3} + \frac{B_2}{(x-3)^2}$$

so that, multiplying both sides of the equality by the denominator of the right hand side,

$$3x^2 - 16x + 6 = A(x-3)^2 + B_1(x+2)(x-3) + B_2(x+2)$$

If we let x take the values

$x = -2:$ $\quad 3(-2)^2 - 16(-2) + 6 = A((-2)-3)^2 + B_1(0) + B_2(0) \;\Rightarrow\; 50 = 25A$

$\Rightarrow A = 2$

$x = 3:$ $\quad 3(3)^2 - 16(3) + 6 = A(0) + B_1(0) + B_2((3)+2) \;\Rightarrow\; -15 = 5B_2$

$\Rightarrow B_2 = -3$

$x = 0:$ $\quad 3(0)^2 - 16(0) + 6 = (2)(-3)^2 + B_1(2)(-3) + (-3)(2) \;\Rightarrow\; 6 = 18 - 6B_1 - 6$

$\Rightarrow B_1 = 1$

Therefore the integral can be written

$$\int \frac{3x^2 - 16x + 6}{(x+2)(x-3)^2}\, dx = \int \left(\frac{2}{x+2} + \frac{1}{x-3} + \frac{-3}{(x-3)^2} \right) dx$$

$$= 2\ln|x+2| + \ln|x-3| + \frac{3}{x-3} + C.$$