Final Exam – Math 2300 – December 16, 2013

On my honor as a University of Colorado at Boulder student I have neither given nor received unauthorized assistance on this exam.

Name:______________________________________

Please select your section:

- ◯ **001** M. GRIMES .... (9 AM)
- ◯ **002** A. SPINA .... (10 AM)
- ◯ **003** A. SPINA .... (11 AM)
- ◯ **004** C. MESA .... (12 NOON)
- ◯ **005** F.T. LIU .... (2 PM)
- ◯ **006** C. BRIDGES .... (3 PM)
- ◯ **007** K. SMITH .... (8 AM)
- ◯ **008** S. HENRY .... (2 PM)

In order to receive full credit your answer must be **complete**, **legible** and **correct**. You should show all of your work, and give clear explanations. **Calculators, phones or other electronic devices are not allowed.**

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	5	
10	5	
11	10	
12	6	
13	10	
14	9	
15	10	
16	8	
17	8	
18	8	
19	10	
20	12	
21	6	
Total:	147	

INSTRUCTIONS FOR MULTIPLE CHOICE: Each of the multiple choice problems is worth five (5) points. No partial credit will be given. You must clearly circle the correct answer. There is only one correct answer per problem. If you circle multiple answers you will receive no credit. You do not need to show your work.

1. (5 points) The value(s) of r for which the function $y(t) = e^{rt}$ is a solution to the differential equation

$$y'' - y' - 2 = 0$$

are:

$$r^2 - r - 2 = 0 \Longrightarrow r = \frac{1 \pm \sqrt{1+8}}{2} = \begin{cases} 2 \\ -1 \end{cases}$$

(a) $r = 0$

(b) $r = 1$

(c) $r = -1$ and $r = 2$

(d) None of the above

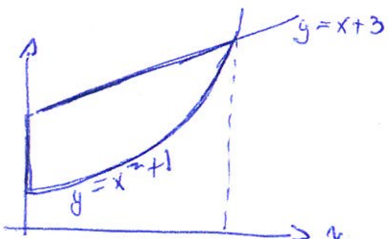
2. (5 points) Let R be the region enclosed by $y = x^2 + 1$, $y = x + 3$, and to the right of $x = 0$. The area of R is given by:

(a) $\int_1^5 \sqrt{y-1} - y + 3\,dy$

(b) $\int_0^2 x^2 + 1 - x - 3\,dx$

(c) $\int_1^2 x + 3 - x^2 - 1\,dx$

(d) $\int_0^2 x + 3 - x^2 - 1\,dx$



$$x + 3 = x^2 + 1$$
$$\Longrightarrow x^2 - \cancel{x} - 2 = 0$$
$$\Longrightarrow x = \begin{cases} 2 \\ -1 \end{cases}$$

$$\int_0^2 \left[x+3 - (x^2+1) \right] dx$$

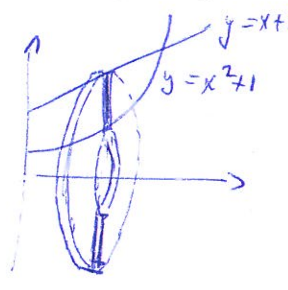
3. (5 points) The volume of the solid obtained after revolving the region R from Problem 2 about the x-axis is represented by:

(a) $\int_0^2 \pi[(x+3)^2 - (x^2+1)^2]\,dx$

(b) $\int_0^2 \pi[(x+3) - (x^2+1)]^2\,dx$

(c) $\int_0^2 \pi[(x^2+1)^2 - (x+3)^2]\,dx$

(d) $\int_0^2 \pi[(x^2+1) - (x+3)]^2\,dx$



4. (5 points) The Taylor series of $g(x) = \sin(x^2)$ near $x = 0$ is

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots.$$

Therefore

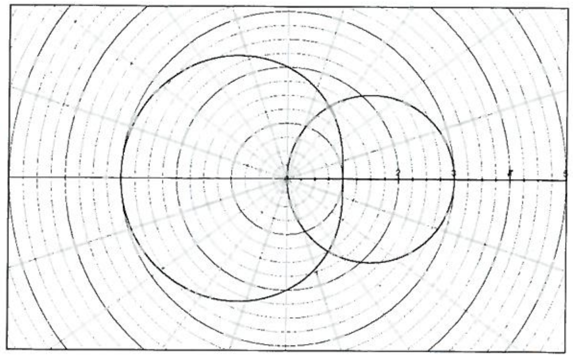
(a) $g''(0) = 1$

(b) $g^{(10)}(0) = \frac{10!}{5!}$

(c) $g^{(n)}(0)$ is not defined for odd values of n

(d) None of the above

5. (5 points) The following is the graph of the polar curves $r = 2 - \cos\theta$ and $r = 3\cos\theta$.



The area of the region inside of $r = 3\cos\theta$ and outside of $r = 2 - \cos\theta$ is given by:

(a) $\frac{1}{2}\int_{-\pi/2}^{\pi/2} 9\cos^2\theta - (2-\cos\theta)^2\, d\theta$

(b) $\frac{1}{2}\int_{-\pi/3}^{\pi/3} 9\cos^2\theta - (2-\cos\theta)^2\, d\theta$

(c) $\frac{1}{2}\int_{-\pi/3}^{\pi/3} (3\cos\theta - (2-\cos\theta))^2\, d\theta$

(d) $\int_{-\pi/3}^{\pi/3} \pi(3\cos\theta - (2-\cos\theta))^2\, d\theta$

6. (5 points) The series $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n\ln n}$ is:

$\int_2^\infty \frac{dx}{x\ln x} = \int_{\ln 2}^\infty \frac{dw}{w} = \infty$

(a) Absolutely convergent
(b) Divergent
(c) Conditionally convergent
(d) Convergent by the limit comparison test

7. (5 points) The series $\sum_{n=1}^{\infty} \frac{\sqrt{k+1}}{k^2+1}$

$k >> 1: \quad \frac{\sqrt{k+1}}{k^2+1} \approx \frac{k^{1/2}}{k^2} = k^{-3/2}$

$\sum_{k=2}^{\infty} \frac{1}{k^{3/2}}$ conv. by p-test

(a) is convergent because $\frac{\sqrt{k+1}}{k^2+1}$ converges to 0.
(b) is absolutely convergent by the ratio test
(c) is convergent by limit comparison
(d) is divergent

8. (5 points) Which of the following is a geometric series?

(a) $2y - 6y^3 + 18y^5 - 54y^7 + \ldots$ $\quad = 2y\left[1 - 3y^2 + 9y^4 - 27y^6 \cdots\right]$

(b) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \ldots$

(c) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$ $\quad = 2y \sum_{j=0}^{\infty} (3y)^{2j}$

(d) $x + x^2 + x^4 + x^7 + \ldots$

9. (5 points) Which of the following has the smallest radius of convergence?

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-10)^n}{6n+1}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = |x-10| \frac{6n+1}{6n+7} \longrightarrow R = 1$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x-10)^n}{n+1}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = |x-10| \frac{n+1}{n+2} \frac{2^{n+1}}{2^n} \longrightarrow R = 1/2$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-10)^n}{n!}$ $\quad = e^{x-10} \longrightarrow R = \infty$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-10)^n}{3^n}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = |x-10| \frac{3^n}{3^{n+1}} \longrightarrow R = 3$

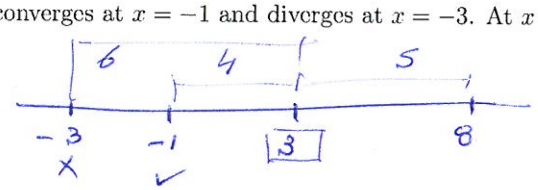
10. (5 points) The power series $\displaystyle\sum_{n=0}^{\infty} a_n(x-3)^n$ converges at $x = -1$ and diverges at $x = -3$. At $x = 8$, the series

(a) Conditionally convergent

(b) Absolutely convergent

(c) Divergent

(d) Cannot be determined



End of multiple choice problems

11. (10 points) Solve the initial value problem

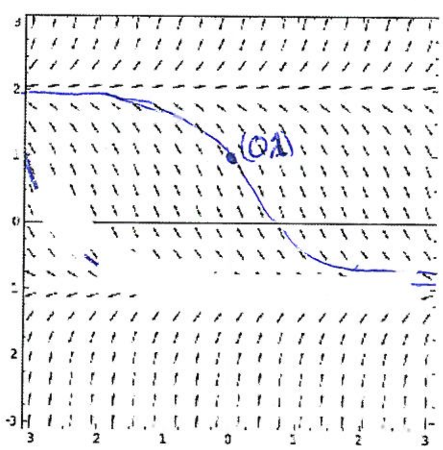
$$\frac{dz}{dt} = z + zt^2, \qquad z(0) = 5$$

$\rightarrow \frac{dz}{dt} = z(1+t^2) \rightarrow \frac{dz}{z} = (1+t^2)\,dt$

$\rightarrow \ln|z| = t + \frac{t^3}{3} + C_1 \rightarrow \boxed{z = C' e^{t + t^3/3}}$

@ $t=0$ $\quad z = 5 = C \rightarrow \boxed{z(t) = 5e^{t+t^3/3}}$

12. (6 points) A slope field for a differential equation $y' = f(x, y)$ is shown.



List any stable equilibrium solution(s) (State "NONE" if there are none.) $y = -1$

List any unstable equilibrium solution(s) (State "NONE" if there are none.) $y = 2$

On the graph, sketch the solution to the differential equation that satisfies $y(0) = 1$.

13. (10 points) Define $f(x,y) = y\ln xy + x^3y^2$. Calculate the following partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{y}{xy}y + 3x^2y^2 = \boxed{\frac{y}{x} + 3x^2y^2}$$

$$\frac{\partial f}{\partial y} = \ln xy + y\frac{1}{xy}x + 2x^3y = \boxed{\ln xy + 1 + 2x^3y}$$

14. (a) (3 points) Find the distance between the point $(2,1,3)$ and the origin.

$$\sqrt{2^2+1^2+3^2} = \sqrt{4+1+9} = \sqrt{14}$$

(b) (3 points) Find the closest distance between the point $(2,1,3)$ and the z-axis.

$$\sqrt{2^2+1^2} = \sqrt{5}$$

(c) (3 points) Find the closest distance between the point $(2,1,3)$ and the xy-plane.

3

15. (10 points) Evaluate the integral of the function $f(x,y) = e^{x+y}$ over the region $0 \le x \le 1$, $0 \le y \le 2$.

$$\int_R e^{x+y}\,dA = \int_R e^x e^y\,dA = \int_{x=0}^{1}\int_{y=0}^{2} e^x e^y\,dy\,dx$$

$$= \left(\int_0^1 e^x dx\right)\left(\int_0^2 e^y dy\right) \quad \text{or} \quad = \int_0^1 \left(e^x e^y \Big|_0^2 \right) dx$$

$$= \boxed{(e-1)(e^2-1)}$$

$$= \int_0^1 e^{x+2} - e^x\,dx$$

$$= e^{x+2} - e^x \Big|_0^1$$

$$= e^3 - e^1 - (e^2 - 1)$$

16. (8 points) Evaluate the integral $\int \frac{\sin(\ln x)}{x}\, dx$

$w = \ln x \longrightarrow dw = \frac{1}{x} dx$

$\therefore \int \frac{\sin(\ln x)}{x} dx \longrightarrow \int \sin(w)\, dw = -\cos w + C$

$\longrightarrow \boxed{-\cos(\ln x) + C}$

17. (8 points) Evaluate the integral $\int \frac{1}{x^2 - x}\, dx$

$\frac{1}{x^2-x} = \frac{1}{x(x-1)} = + \frac{A}{x} + \frac{B}{x-1} \cdots\cdots \longrightarrow \begin{cases} A = -1 \\ B = 1 \end{cases}$

$\int \frac{dx}{x^2-x} = \int \frac{dx}{x(x-1)} = \int \left\{ -\frac{1}{x} + \frac{1}{x-1} \right\} dx = -\ln|x| + \ln|x-1| + C$

$= \boxed{\ln \left| \frac{x-1}{x} \right| + C}$

18. (8 points) Evaluate the integral $\int x \ln x\, dx$ $= \int \underbrace{(\ln x)}_{u} \underbrace{(x\, dx)}_{dv}$

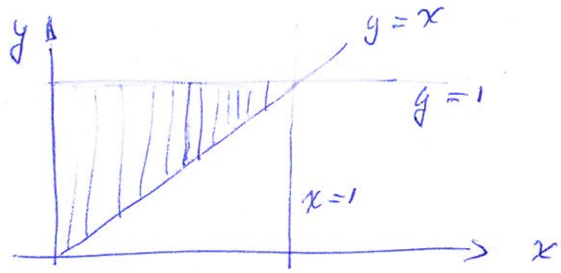
$= \frac{x^2}{2} \ln(x) - \int \frac{x^2}{2} \frac{1}{x} dx$

$= \boxed{\frac{x^2}{2} \ln(x) - \frac{x^2}{4} + C}$

19. (10 points) For the double integral

$$\int_0^1 \int_x^1 \frac{y^2}{1+y^4}\,dy\,dx$$

(a) Sketch the region of integration.



(b) Evaluate the integral.

$$\int_{x=0}^{1} \int_{y=x}^{1} \frac{y^2}{1+y^4}\,dy\,dx = \int_{y=0}^{1} \int_{x=0}^{y} \frac{y^2}{1+y^4}\,dx\,dy$$

$$= \int_{y=0}^{1} \frac{y^2}{1+y^4}\left(\int_{x=0}^{y} dx\right) dy = \int_{y=0}^{1} \frac{y^2}{1+y^4}\, y\,dy$$

$$= \int_0^1 \frac{y^3\,dy}{1+y^4} = \left\{\frac{1}{4} \ln\left[y^4+1\right]\right\}_{y=0}^{1} = \boxed{\frac{1}{4}\ln 2}$$

20. (a) (6 points) Find the Taylor series about $x = 0$ for $f(x) = xe^x - x$.

$$xe^x - x = x \sum_{k=0}^{\infty} \frac{x^k}{k!} - x$$

$$= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!} - x$$

$$= \boxed{\sum_{k=1}^{\infty} \frac{x^{k+1}}{k!}}$$

(b) (6 points) Find the 4th degree Taylor polynomial for $g(x) = (\sin x)^2$ about $x = 0$.

$$(\sin x)^2 = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2$$

$$= x^2 - 2x\frac{x^3}{3!} + \cdots$$

$$= x^2 - \frac{x^4}{3} \cdots$$

$$P_4(x) = x^2 - \frac{x^4}{3}$$

21. (6 points) Consider the surface defined by $z = 4x^2 + 4y^2 - 1$

(a) What is the equation of the intersection of the surface with the xy-plane?

$$\begin{cases} z = 4x^2 + 4y^2 - 1 \\ z = 0 \end{cases} \Rightarrow \boxed{4x^2 + 4y^2 = 1}$$

circle $C(0,0)$, $R = \frac{1}{2}$

(b) What is the equation of the intersection of the surface with the xz-plane?

$$\begin{cases} z = 4x^2 + 4y^2 - 1 \\ y = 0 \end{cases} \Rightarrow \boxed{z = 4y^2 - 1}$$

parabola