Section 6.4: Second Fundamental Theorem of Calculus

Let $f(x)$ be a function defined on an interval $I$. Suppose we want to find an antiderivative $F(x)$ of $f(x)$ on the interval $I$. Sometimes, we are able to find an expression for $F(x)$ analytically. For example, if $f(x) = x^2$, then we can take $F(x) = \frac{x^3}{3}$. However, there are elementary functions $f(x)$ (functions that are combinations of constants, powers of $x$, $\sin x$, $\cos x$, $e^x$, and $\ln x$) that do not have an antiderivative $F(x)$ that can be expressed as an elementary function. One such example of an elementary function that does not have an elementary antiderivative is $f(x) = \sin(x^2)$.

The Second Fundamental Theorem of Calculus studied in this section provides us with a tool to construct antiderivatives of continuous functions, even when the function does not have an elementary antiderivative:

**Second Fundamental Theorem of Calculus.** Let $f$ be a continuous function defined on an interval $I$. Fix a point $a$ in $I$ and define a function $F$ on $I$ by

$$F(x) = \int_a^x f(t)dt.$$ 

Then $F$ is an antiderivative of $f$ on the interval $I$, i.e. $F'(x) = f(x)$ on $I$.

A proof of the Second Fundamental Theorem of Calculus is given on pages 318–319 of the textbook.

We note that $F(x) = \int_a^x f(t)dt$ means that $F$ is the function such that, for each $x$ in the interval $I$, the value of $F(x)$ is equal to the value of the integral $\int_a^x f(t)dt$. Furthermore, $F(a) = \int_a^a f(t)dt = 0$, and so $F$ is the antiderivative of $f$ that satisfies $F(a) = 0$.

Now since $\int_a^x f(t)dt$ is an antiderivative of $f(x)$, then the general form of an antiderivative of $f(x)$ is given by

$$F(x) = C + \int_a^x f(t)dt,$$

where $C$ is a constant. In this case, we compute

$$F(a) = C + \int_a^a f(t)dt = C + 0 = C.$$

Therefore we have the result that the general form of an antiderivative of $f(x)$ is given by

$$F(x) = C + \int_a^x f(t)dt, \text{ where } C = F(a).$$

We also note that the fact that $\int_a^x f(t)dt$ is an antiderivative of $f(x)$ in the Second Fundamental Theorem of Calculus can be expressed as

$$\frac{d}{dx} \int_a^x f(t)dt = f(x).$$
Example 1. Let \(f(x) = \sin(x^2)\). Then the function

\[F(x) = \int_0^x \sin(t^2)dt\]

is the antiderivative of \(f\) that satisfies \(F(0) = 0\). For every real number \(x\), we can find the value of \(F(x)\) by computing numerically the integral \(\int_0^x \sin(t^2)dt\). We give a few values of \(F\) in the table below:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F(x))</td>
<td>-0.7736</td>
<td>-0.8048</td>
<td>-0.3103</td>
<td>0</td>
<td>0.3103</td>
<td>0.8048</td>
<td>0.7736</td>
</tr>
</tbody>
</table>

We remark that the table suggests that \(F\) is an odd function, i.e. \(F(-x) = -F(x)\). Indeed, since the function \(f(t) = \sin(t^2)\) is even (as \(f(-t) = \sin((-t)^2) = \sin(t^2) = f(t)\)), we must have, following a result from Section 5.4, that

\[\int_{-x}^0 \sin(t^2)dt = \int_x^0 \sin(t^2)dt.\]

From here,

\[F(-x) = \int_0^{-x} \sin(t^2)dt = -\int_0^0 \sin(t^2)dt = -\int_0^x \sin(t^2)dt = -F(x),\]

and so \(F\) is an odd function.

Example 2. Suppose we know that the function \(f(x)\) is such that \(f'(x) = e^{-x^2}\) and \(f(0) = 2\). Then an expression for \(f(x)\) is given by

\[f(x) = 2 + \int_0^x e^{-t^2}dt.\]

If we are asked to find the value of \(f(3)\), we have

\[f(3) = 2 + \int_0^3 e^{-t^2}dt = 2 + 0.8862 = 2.8862,\]

where the above integral is computed numerically.

Example 3.

(a) Find \(\frac{d}{dx} \int_{\frac{1}{2}}^x \ln(t^2 + 1)dt\).

A direct application of the Second Fundamental Theorem of Calculus yields

\[\frac{d}{dx} \int_{\frac{1}{2}}^x \ln(t^2 + 1)dt = \ln(x^2 + 1).\]

(b) Find \(\frac{d}{dt} \int_{\frac{1}{t}}^\pi \cos(z^3)dz\).

First, we need to switch the limits of integration, and then we apply the Second Fundamental Theorem of Calculus:

\[\frac{d}{dt} \int_{\frac{1}{t}}^\pi \cos(z^3)dz = \frac{d}{dt} \left( -\int_\pi^t \cos(z^3)dz \right) = -\frac{d}{dt} \int_\pi^t \cos(z^3)dz = -\cos(t^3).\]
(c) Find \( \frac{d}{dx} \int_{x^2}^{x^3} \sin(t^2)dt \).

Let \( G(x) = \int_{2}^{x^2} \sin(t^2)dt \). Then, by the Second Fundamental Theorem of Calculus,

\[
G'(x) = \frac{d}{dx} \int_{2}^{x^2} \sin(t^2)dt = \sin(x^2).
\]

Since \( \int_{2}^{x^3} \sin(t^2)dt = G(x^3) \), we are asked to find \( \frac{d}{dx} \left(G(x^3)\right) \). By the chain rule,

\[
\frac{d}{dx} \left(G(x^3)\right) = 3x^2G'(x^3).
\]

Hence

\[
\frac{d}{dx} \int_{x^2}^{x^3} \sin(t^2)dt = \frac{d}{dx} \left(G(x^3)\right) = 3x^2 \sin(x^6).
\]

(d) Find \( \frac{d}{dt} \int_{t^2}^{\cos t} \sqrt{1 + x^4}dx \).

We start by breaking up the integral in two and then switching the limits of integration in the first integral:

\[
\int_{t^2}^{\cos t} \sqrt{1 + x^4}dx = \int_{t^2}^{0} \sqrt{1 + x^4}dx + \int_{0}^{\cos t} \sqrt{1 + x^4}dx = \int_{0}^{t^2} \sqrt{1 + x^4}dx + \int_{0}^{\cos t} \sqrt{1 + x^4}dx.
\]

Let \( G(t) = \int_{0}^{t} \sqrt{1 + x^4}dx \). Then we can write

\[
\int_{t^2}^{\cos t} \sqrt{1 + x^4}dx = - \int_{0}^{t^2} \sqrt{1 + x^4}dx + \int_{t^2}^{\cos t} \sqrt{1 + x^4}dx = -G(t^2) + G(\cos t).
\]

By the chain rule,

\[
\frac{d}{dt} \int_{t^2}^{\cos t} \sqrt{1 + x^4}dx = \frac{d}{dt} \left(-G(t^2) + G(\cos t)\right) = -2tG'(t^2) - (\sin t)G'(\cos t).
\]

By the Second Fundamental Theorem of Calculus, we have

\[
G'(t) = \frac{d}{dt} \int_{0}^{t} \sqrt{1 + x^4}dx = \sqrt{1 + t^4}.
\]

Hence

\[
\frac{d}{dt} \int_{t^2}^{\cos t} \sqrt{1 + x^4}dx = -2tG'(t^2) - (\sin t)G'(\cos t)
\]

\[
= -2t\sqrt{1 + (t^2)^4} - (\sin t)\sqrt{1 + (\cos t)^4} = -2t\sqrt{1 + t^8} - (\sin t)\sqrt{1 + (\cos t)^4}.
\]