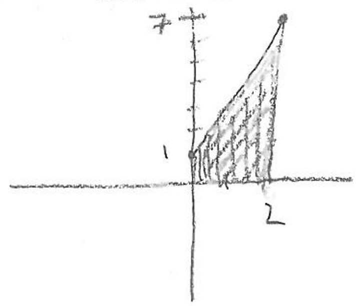
CALCULATING A DEFINITE INTEGRAL FROM THE LIMIT OF A RIEMANN SUM.

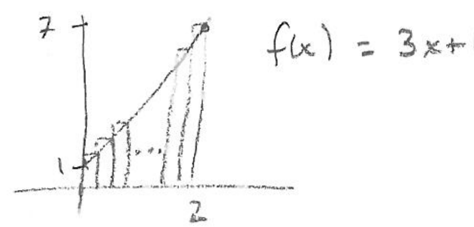
<u>EXAMPLE:</u> EVALUATE $\int_0^2 3x+1\,dx$, USING THE LIMIT OF RIGHT RIEMANN SUMS.

THIS INTEGRAL CORRESPONDS TO THE AREA OF THE SHADED REGION:



(NOTE: FROM GEOMETRY, THE AREA IS 8, SO IN THIS EXAMPLE, WE ALREADY KNOW THE ANSWER BY ANOTHER METHOD)

- SLICE IT INTO n RECTANGLES



WIDTH OF RECTANGLES $\Delta x = \frac{2}{n}$

- RIGHT-HAND ENDPOINTS:

$$\frac{2}{n}, \frac{4}{n}, \frac{6}{n}, \ldots \frac{2n}{n}$$

(THESE ARE GIVEN BY THE FORMULA $x_i = \frac{2i}{n}$)

- HEIGHT OF RECTANGLES: PLUG RIGHT-HAND ENDPOINTS INTO $f(x) = 3x+1$:

$$3\left(\frac{2}{n}\right)+1, \quad 3\left(\frac{4}{n}\right)+1, \quad 3\left(\frac{6}{n}\right)+1, \quad \ldots \quad 2\left(\frac{2n}{n}\right)+1$$

(THESE HEIGHTS ARE GIVEN BY $f(x_i) = 3\cdot\left(\frac{2i}{n}\right)+1$)

- AREA OF RECTANGLES: (height • width)

$$\left(3\left(\frac{2}{n}\right)+1\right)\left(\frac{2}{n}\right), \left(3\left(\frac{4}{n}\right)+1\right)\left(\frac{2}{n}\right), \ldots \left(3\left(\frac{2n}{n}\right)+1\right)\left(\frac{2}{n}\right)$$

(THESE AREAS ARE GIVEN BY $f(x_i)\Delta x = \left(3\left(\frac{2i}{n}\right)+1\right)\left(\frac{2}{n}\right)$)

- RIEMANN SUM:

$$\left(3\left(\frac{2}{n}\right)+1\right)\left(\frac{2}{n}\right) + \left(3\left(\frac{4}{n}\right)+1\right)\left(\frac{2}{n}\right) + \ldots + \left(3\left(\frac{2n}{n}\right)+1\right)\left(\frac{2}{n}\right)$$

(IN SUMMATION NOTATION: $\sum_{i=1}^{n} \left(3\left(\frac{2i}{n}\right)+1\right)\left(\frac{2}{n}\right)$)

- SIMPLIFY THE RIEMANN SUM:

$$\left(\frac{6}{n}+1\right)\left(\frac{2}{n}\right) + \left(\frac{12}{n}+1\right)\left(\frac{2}{n}\right) + \ldots + \left(\frac{6n}{n}+1\right)\left(\frac{2}{n}\right)$$

$$= \left(\frac{6}{n}\cdot\frac{2}{n} + \frac{2}{n}\right) + \left(\frac{12}{n}\cdot\frac{2}{n} + \frac{2}{n}\right) + \ldots + \left(\frac{6n}{n}\cdot\frac{2}{n} + \frac{2}{n}\right)$$

$$= \left(\frac{12}{n^2} + \frac{24}{n^2} + \dots \quad \frac{12n}{n^2}\right) + \left(\frac{2}{n} + \frac{2}{n} + \dots + \frac{2}{n}\right)$$

$$= \frac{12}{n^2}\left(1+2+\dots+n\right) + n \cdot \frac{2}{n}$$

NOTE THAT $1+2+\dots+n = \frac{n(n+1)}{2}$

$$= \frac{12}{n^2} \cdot \frac{n(n+1)}{2} + 2$$

$$= \frac{12}{2} \cdot \frac{n^2+n}{n^2} + 2$$

$$= 6 \cdot \left(1+\frac{1}{n}\right)+2 = 8 + \frac{6}{n}$$

NOTE: THIS ENTIRE SIMPLIFICATION CAN ALTERNATELY BE DONE IN SIGMA NOTATION:

$$\sum_{i=1}^{n} \left(3\left(\frac{2i}{n}\right)+1\right)\frac{2}{n}\right) = \sum_{i=1}^{n} \left(\frac{6i}{n}+1\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} \left(\frac{12i}{n^2} + \frac{2}{n}\right) = \sum_{i=1}^{n} \frac{12i}{n^2} + \sum_{i=1}^{n} \frac{2}{n}$$

$$= \frac{12}{n^2} \sum_{i=1}^{n} i + \frac{2}{n}\sum_{i=1}^{n} 1$$

Note: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} 1 = n$

$$= \frac{12}{n^2} \cdot \frac{n(n+1)}{2} + \frac{2}{n} \cdot n$$

$$= 6 \cdot \frac{n^2+n}{n^2} + 2$$

$$= 6\left(1+\frac{1}{n}\right)+2 = 8 + \frac{6}{n}$$

- TAKE THE LIMIT OF THE RIEMANN SUMS:

$$\lim_{n\to\infty} 8 + \frac{6}{n} = 8$$

SO $\int_0^2 3x+1 \, dx = 8$