**Review problems for Final Exam**
**Mathematics 1300, Calculus 1** – Solutions

1. For each part, find $dy/dx$.

   (a) $y = \dfrac{3x^3 + 2x^2}{\sqrt{x}}$

   **Answer:** $\frac{dy}{dx} = \frac{15}{2}x^{3/2} + 3x^{1/2}$

   (b) $y = xe^{2x}$

   **Answer:** $\frac{dy}{dx} = (2x+1)e^{2x}$

   (c) $y = \frac{1}{2}\sin(-2x)$

   **Answer:** $\frac{dy}{dx} = -\cos(-2x)$

   (d) $xy = x^2 + y^2$

   **Answer:** $\frac{dy}{dx} = \frac{2x-y}{x-2y}$

   (e) $y = \displaystyle\int_3^x e^{-t^2}\,dt$

   **Answer:** $\frac{dy}{dx} = e^{-x^2}$

   (f) $y = (-5x+2)^4$

   **Answer:** $\frac{dy}{dx} = -20(-5x+2)^3$

   (g) $y = \int_x^2 3\sin(t^2)\,dt$.

   **Answer:** $-3\sin(x^2)$

   (h) $\dfrac{d}{dx}[\pi^e + 2]$

   **Answer:** 0. It's just the derivative of some complicated constant.

   (i) $\dfrac{d}{dx}\left[3^x + x^3\right]$

   **Answer:** $3^x \ln 3 + 3x^2$

   (j) $\dfrac{d}{dt}[\arctan(t)\ln(t/5)]$

   **Answer:** $\dfrac{\ln(t/5)}{1+t^2} + \dfrac{\arctan t}{t}$

   (k) $\dfrac{d}{dy}\left[\sqrt{\dfrac{y+1}{y+7}}\right]$

   **Answer:** $3\sqrt{\dfrac{y+7}{y+1}}\,\dfrac{1}{(y+7)^2}$

   (l) Let $y = \displaystyle\int_{2x}^{x^3} \frac{8}{\ln(4t)}\,dt$, for $x > 0$.

   **Answer:**

$$\frac{dy}{dx} = \frac{8\cdot 3x^2}{\ln(4x^3)} - \frac{8\cdot 2}{\ln(4\cdot 2x)}.$$

2. Find the most general antiderivative.

   (a) $\displaystyle\int \frac{1}{x^3}\,dx$

   **Answer:** $-\frac{1}{2}x^{-2} + C$

   (b) $\displaystyle\int \frac{1}{\cos^2 x}\,dx$

   **Answer:** $\tan x + C$

   (c) $\displaystyle\int (3e^{2x} + 2\sin x)\,dx$

   **Answer:** $\frac{3}{2}e^{2x} - 2\cos x + C$ .

   (d) $\displaystyle\int (x+1)^9\,dx$

   **Answer:** $\frac{(x+1)^{10}}{10} + C$

   (e) $\displaystyle\int \frac{x^2+x+1}{x}\,dx$

   **Answer:** $\int (x + 1 + \frac{1}{x})dx = \frac{1}{2}x^2 + x + \ln|x| + C$

3. Evaluate the following integrals

   (a) $\displaystyle\int_0^2 \sqrt{4-y^2}\,dy$

   **Answer:** You're supposed to recognize this as the area of a quarter-circle of radius 2, which is $\frac{1}{4}(\pi \cdot 2^2) = \pi$.

   (b) $\displaystyle\int \frac{t+\sqrt{t}+1}{t^2}\,dt$

   **Answer:** $\displaystyle\int \frac{t+\sqrt{t}+1}{t^2}\,dt = \int \left(t^{-1} + t^{-3/2} + t^{-2}\right)\,dt = \ln|t| - 2t^{-1/2} - t^{-1} + C$

   (c) $\displaystyle\int_1^2 \left[2^\theta + 1\right]\,d\theta$

   **Answer:** It's $\displaystyle\left(\frac{2^\theta}{\ln 2} + \theta\right)\Big|_{\theta=1}^{\theta=2} = \left(\frac{2^2}{\ln 2} + 2\right) - \left(\frac{2^1}{\ln 2} + 1\right) = \frac{2}{\ln 2} + 1.$

   (d) $\displaystyle\int \sin x \cos^2 x\,dx$

   **Answer:** $\displaystyle -\frac{1}{3}\cos^3 x + C$

   (e) $\displaystyle\int_0^1 x(x^2+1)^4\,dx$

   **Answer:** $\frac{1}{10}(x^2+1)^5\Big|_0^1 = \frac{32-1}{10} = \frac{31}{10}$

4. Let $C(q)$ be the total cost of producing $q$ lawnmowers. Which of the following gives the meaning of $C'(1000)$?

(a) The cost of producing 1001 lawnmowers.

(b) The average cost of producing each of the first 1000 lawnmowers.

(c) The approximate cost of producing the 1001st lawnmower.

(d) None of the above.

**Answer:** $C'(1000) \approx C(1001) - C(1000)$, so it represents the additional cost to produce one more lawnmower when 1000 are already being produced. The answer is (c).

5. Suppose $f(x)$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Which of the following is true?

(a) If $f(x)$ has a critical point at $x = 0$ and $f''(x) < 0$, then $f(x)$ has a local minimum at $x = 0$.

(b) If $f(x)$ has a critical point at $x = 0$ and $f''(x) > 0$, then $f(x)$ has a local minimum at $x = 0$.

(c) If $f(x)$ has a critical point at $x = 0$, then $f(x)$ has a local minimum or maximum at $x = 0$.

(d) If $f(x)$ has a critical point at $x = 0$ and $f''(0) = 0$, then $x = 0$ is an inflection point.

**Answer:** Only (b) is true, by the Second Derivative Test. (c) is false by considering $f(x) = x^3$, and (d) is false by considering $f(x) = x^4$.

6. Suppose that $F(x) = \displaystyle\int_1^x \ln t \, dt$ for $x > 0$. Which of the following statements is false?

(a) $F(1) = 0$.

(b) $F'(e) = 1$.

(c) $F(x)$ is increasing at $x = 2$.

(d) $F(x)$ is increasing at $x = \frac{1}{2}$.

**Answer:** By substituting $x = 1$, the integral is 0, so (a) is true. By the Fundamental Theorem of Calculus, $f'(x) = \ln x$, so $F'(e) = \ln e = 1$, so (b) is true. $F'(2) = \ln 2$, which is positive, so (c) is true. $F'(x) = \ln x$, and we have $F'(\frac{1}{2}) = -\ln 2 < 0$. So (d) is false.

7. What are the inflection points of $f(x) = x^5 - 5x^4 - 50$?

**Answer:** $f''(x) = 20x^3 - 60x^2 = 20x^2(x - 3)$. The second derivative is 0 at $x = 0$ and $x = 3$. A sign chart shows the the second derivative does not change signs at $x = 0$ but it does change signs at $x = 3$. So $x = 3$ is the only inflection point.

8. Solve the differential equation $\dfrac{dq}{dz} = 2 + \sin(z)$ with initial condition $q(\pi) = 2\pi$.

**Answer:** $q(z) = 2z - \cos z - 1$

9. Using the left-hand Riemann sum with $n = 4$, approximate $\int_1^9 \frac{1}{x}\,dx$.

**Answer:**

$$\int_1^9 \frac{1}{x}\,dx \approx 2 \cdot \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}\right) = \frac{352}{105}.$$

10. Suppose that $f(2) = 4$, and that the table below gives values of $f'$ for $x$ in the interval $[0, 12]$

| $x$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | $-19$ | $-21$ | $-25$ | $-28$ | $-29$ | $-28$ | $-25$ |

Estimate $f''(2)$, and estimate $f(8)$.

**Answer:** $f''(2) \approx \dfrac{f'(4) - f'(0)}{4 - 0} = \dfrac{-25 + 19}{4} = -1.5$. By the Fundamental Theorem of Calculus, $f(8) = f(2) + \int_2^8 f'(x)dx = 4 + \int_2^8 f'(x)dx$. Estimating the integral using the trapezoid rule gives $-156$, so $f(8) \approx 4 - 156 = -152$.

11. If $H(3) = 1$, $H'(3) = 3$, $F(3) = 5$, $F'(3) = 4$, find $G'(3)$ if $G(w) = F(w)/H(w)$.

**Answer:** $G'(3) = \dfrac{H(3)F'(3) - F(3)H'(3)}{H(3)^2} = \dfrac{1 \cdot 4 - 5 \cdot 3}{1^2} = -11$.

12. What is the largest area a rectangle with a perimeter of 40 inches can have?

**Answer:** Set up $2x + 2y = 40$ and $A = xy$ to get $A = x(20 - x)$. The domain for $x$ is $[0, 20]$, because $x$ cannot be negative, and neither can $y$. $A'(x) = 20 - 2x$, which is zero when $x = 10$. When $x = 0$ or $x = 20$ the area is zero, so the maximum occurs at the critical point, when $x = 10$. The maximum area is $A = 100$.

13. A rocket's height (in feet) is given by $s(t) = 3e^{2t} + 10$, where $t$ is in seconds. How fast is the rocket traveling when it reaches a height of 40 feet?

**Answer:** It reaches 40 feet when $t = \frac{1}{2}\ln 10$, and at this time $v(t) = s'(t) = 6e^{2t} = 60$ feet per second.

14. A circular oil spill is growing. Its radius is increasing at a rate of 10 feet per minute. When the radius is 2500 feet, at what rate is the area of the oil spill growing?

**Answer:** $A = \pi r^2$, so $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. Substituting $r = 2500$ feet and $\frac{dr}{dt} = 10$ feet per minute gives $\frac{dA}{dt} = 50000\pi$ square feet per minute.

15. Assume $f'$ is given by the graph below.

a) Find the value of the integral: $\int_0^7 f'(x)\,dx$

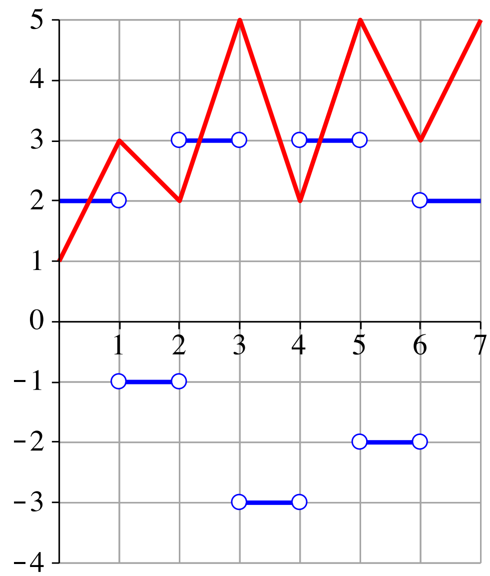
**Answer:** $\int_0^7 f'(x)\,dx = 2 - 1 + 3 - 3 + 3 - 2 + 2 = 4.$

b) What is the second derivative of $f$, i.e., $f''$?

**Answer:** It is zero everywhere it exists, and it does not exist at any of the integers.

c) Suppose $f$ is continuous and that $f(0) = 1$. Sketch an accurate graph of $f$ in the above box (which already contains a graph of $f'$ ).

**Answer:** It's piecewise linear, so we just need to fill it in at the integers and then connect the dots.




16. Represent the finite area enclosed by the two curves $y = x^2$ and $y = \sqrt{x}$ as an integral and evaluate it.

**Answer:** The curves cross at $x = 0$ and $x = 1$, and $y = \sqrt{x}$ is on top, so the area is

$$A = \int_0^1 \sqrt{x} - x^2\,dx = \frac{1}{3}.$$

17. You have 80 feet of fencing and want to enclose a rectangular area up against a long, straight wall (using the wall for one side of the enclosure and the fencing for the other three sides of the enclosure). What is the largest area you can enclose?

**Answer:** Letting $x$ be the length of the side perpendicular to the wall and $y$ be the length of the side parallel to the wall, we have $2x + y = 80$ and $A = xy$ to get $A = x(80 - 2x)$. The domain is that $x$ must be in the interval $[0, 40]$. Taking the derivative of $A(x)$ and setting it equal to 0 gives a critical point at $x = 20$. Evaluating the area at the endpoints and the critical point gives a maximum area of 800 square feet.

18. Find the global maximum and minimum for $h(x) = \dfrac{x+1}{x^2+3}$ on the interval $-1 \leq x \leq 2$.

**Answer:** Critical points occur at $x = 1$ and $x = -3$; only $x = 1$ is relevant since it's in the interval. Test $h(-1) = 0$, $h(1) = \frac{1}{2}$, and $h(2) = \frac{3}{7}$ to find the global minimum at $x = -1$ and the global maximum at $x = 1$.

19. Fill in each of the blanks below with the best possible answer.

a) If $f$ is differentiable on $a \leq x \leq b$, then there exists a number c, with $a < c < b$, such that

$$f'(c) = \underline{\frac{f(b) - f(a)}{b - a}}.$$

b) For any function $f$, the function whose value at $x$ is given by

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

is called $f'(x)$, or the derivative of $f$ .

20. Evaluate each of the following limits.

a) $\displaystyle\lim_{x\to 0}\frac{2^{-x}-x}{x^2+9}$

**Answer:** Plug in $x=0$ to get $1/9$.

b) $\displaystyle\lim_{x\to\infty}\frac{3x^2-x^4-5x^5}{x^5-1}$

**Answer:** Either multiply numerator and denominator by $\frac{1}{x^2}$ and simplify, or use L'Hopital's rule to get $-5$.

c) $\displaystyle\lim_{x\to 0^-}\frac{|x-1|}{x-1}$

**Answer:** Plug in $x=0$ to get $-1$.

d) $\displaystyle\lim_{x\to 1^-}\frac{|x-1|}{x-1}$

**Answer:** As $x$ approaches 1 from the left, $|x-1|=1-x$. Simplifying and substituting gives $-1$.

e) $\displaystyle\lim_{x\to\infty} x^2e^{-x}$

**Answer:** Use L'Hopital's rule to get

$$\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}=\lim_{x\to\infty}\frac{2}{e^x}=0.$$

f) $\displaystyle\lim_{x\to 0^+} x\ln x$

**Answer:** It's of the form $0\cdot\infty$, so we rewrite as a fraction:

$$\lim_{x\to 0^+} x\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0^+}(-x)=0.$$

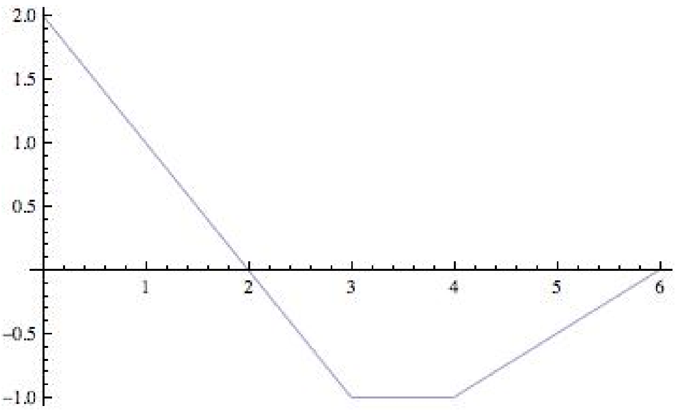
21. Find the equation of the tangent line to the curve $y=\sqrt{25-x^2}$ where $x=3$.

**Answer:** $y-4=-\frac{3}{4}(x-3)$.

22. Let $f(x)$ be the function whose graph is given below. Define

$$F(x)=\int_0^x f(t)dt.$$

Use the graph to fill in the entries in the table below:




| x | 2 | 3 |
|---|---|---|
| $F(x)$ | 2 | 1.5 |
| $F'(x)$ | 0 | $-1$ |
| $F''(x)$ | $-1$ | DNE |

23. A 10-meter ladder is leaning against the wall of a building. The base of the ladder begins sliding away from the building at a rate of 3 meters per second. How fast is the top of the ladder sliding down the wall when the base of the ladder is 6 meters from the wall?

    **Answer:** Differentiate $x^2 + y^2 = 10^2$ with respect to time to get $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$. When $x = 6$, $y = 8$. Plug in $x = 6$ and $\frac{dx}{dt} = 3$ and $y = 8$ to get $\frac{dy}{dt} = -\frac{9}{4}$ meters per second.

24. If $f(x)$ is a function having all four of the following properties:

$$\int_0^1 f'(x)\,dx = 2, \quad \int_1^2 f'(x)\,dx = 4, \quad \int_2^3 f'(x)\,dx = 8, \quad f(3) = 16,$$

    then determine the following:

    (a) $\displaystyle\int_0^3 f'(x)\,dx$

    **Answer:** $\displaystyle\int_0^3 f'(x)\,dx = \int_0^1 f'(x)\,dx + \int_1^2 f'(x)\,dx + \int_2^3 f'(x)\,dx = 2+4+8 = 14.$

    (b) $f(0)$.

    **Answer:** $\displaystyle f(3) - f(0) = \int_0^3 f'(x)\,dx = 14$. So $16 - f(0) = 14$, and $f(0) = 2$.

25. Let

$$F(x) = \int_0^{x^2} e^{-4t^2}\,dt.$$

    (a) Find $F'(x)$.

    **Answer:** $F'(x) = 2xe^{-4x^4}$

    (b) Find $F''(x)$.

    **Answer:** $F''(x) = 2e^{-4x^4} - 32x^4e^{-4x^4}$

    (c) Find the $x$ coordinates of all inflection points of $F(x)$.

    **Answer:** $F''(x) = 0$ when $2 = 32x^4$ or $x^4 = \frac{1}{16}$, so $x = \pm\frac{1}{2}$. These are both genuine inflection points since $F''$ changes sign: $F''(-1) < 0$, $F''(0) > 0$, and $F''(1) < 0$.

26. Find the area between the graphs of $f(x) = x + 6$ and $g(x) = x^2$.

**Answer:** The graphs cross when $x + 6 = x^2$, i.e., $x = 3$ and $x = -2$. The line is above the parabola in this region, so the area is

$$A = \int_{-2}^{3} (x + 6 - x^2)\,dx = \left(\frac{1}{2}x^2 + 6x - \frac{1}{3}x^3\right)\Big|_{x=-2}^{x=3} = \frac{27}{2} - \left(-\frac{22}{3}\right) = \frac{125}{6}.$$

27. State the definition of the derivative of $f(x)$ at $x = a$.

**Answer:** $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$

28. Find the following derivatives using the definition of derivative (as a limit of difference quotients). Note: For this problem you may use shortcuts (derivative formulas) to **check** your solution. However, to get full credit for this problem you need to use the **definition of the derivative** to find $f'(x)$.

(a) $f(x) = \frac{2}{x^2}$.
**Answer:**

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{2}{(x+h)^2} - \frac{2}{x^2}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{2x^2}{x^2(x+h)^2} - \frac{2(x+h)^2}{x^2(x+h)^2}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{2x^2 - 2(x+h)^2}{x^2(x+h)^2}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{2x^2 - 2x^2 - 4xh - 2h^2}{x^2(x+h)^2}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{-4xh - 2h^2}{x^2(x+h)^2}\right) \\
&= \lim_{h\to 0} \frac{-4x - 2h}{x^2(x+h)^2} \\
&= \frac{-4x - 2*0}{x^2(x+0)^2} = \frac{-4x}{x^4} = \frac{-4}{x^3}
\end{aligned}$$

(b) $g(x) = \sqrt{x}$.

**Answer:**

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})} \\
&= \frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{2\sqrt{x}}
\end{aligned}$$

(c) $h(x) = 3x^2$.

**Answer:**

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{3(x+h)^2-3x^2}{h} \\
&= \lim_{h\to 0} \frac{3x^2+6xh+3h^2-3x^2}{h} \\
&= \lim_{h\to 0} \frac{6xh+3h^2}{h} \\
&= \lim_{h\to 0} (6x+3h) \\
&= 6x.
\end{aligned}$$

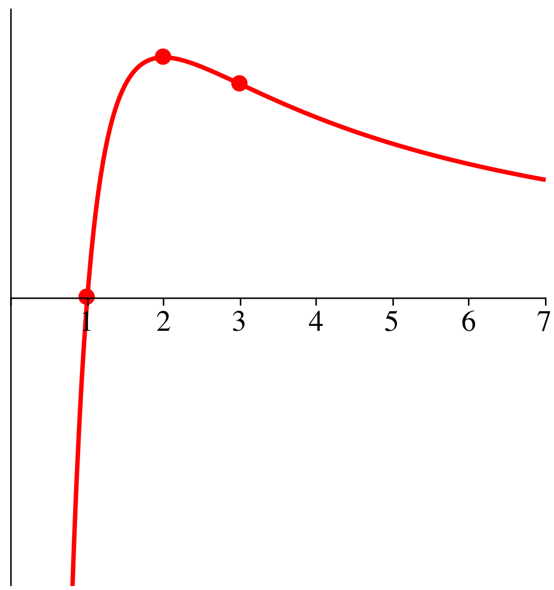
29. Sketch the graph of a continuous, differentiable function $f$, whose domain is $0 < x < \infty$, given the following information about $f$, $f'$, and $f''$:

$$\lim_{x\to 0^+} f(x) = -\infty,$$

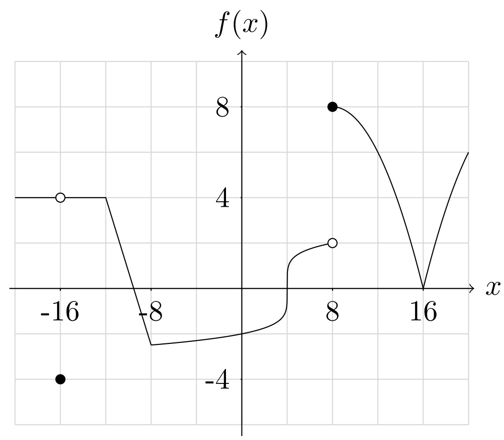
$$\begin{array}{ccc}
f(1) = 0, & f(x) < 0 \text{ for } x < 1, & f(x) > 0 \text{ for } x > 1, \\
f'(2) = 0, & f'(x) > 0 \text{ for } x < 2, & f'(x) < 0 \text{ for } x > 2, \\
f''(3) = 0, & f''(x) < 0 \text{ for } x < 3, & f''(x) > 0 \text{ for } x > 3,
\end{array}$$

$$\lim_{x\to +\infty} f(x) = 0.$$

**Answer:**




30. The graph of the function $f(x)$ is given below:




List all $x$-values at which $f(x)$ is **not** continuous, and list all $x$-values at which $f(x)$ is **not** differentiable.

**Answer:** $f$ is not continuous at $x = -16$ and $x = 8$. $f$ is not differentiable at $x = -16, -12, -8, 4, 8, 16$.

31. Assuming that $f(1) = 9$ and $f'(1) = 4$, find

(a) $g'(1)$, where $g(x) = \sqrt{f(x)}$.
**Answer:** $g'(1) = \frac{1}{2}f(1)^{-1/2}f'(1) = \frac{2}{3}$.

(b) $h'(1)$, where $h(x) = f(\sqrt{x})$.
**Answer:** $h'(1) = f'(\sqrt{1}) \cdot \frac{1}{2}1^{-1/2} = 2$.

32. Estimate $\sqrt[3]{8.1}$ using linear approximation.

**Answer:** Let $f(x) = \sqrt[3]{x}$, $a = 8$, $x = 8.1$. Note that $f'(x) = \frac{1}{3}x^{-2/3}$. And $f(8) = 2, f'(8) = \frac{1}{3}8^{-2/3} = \frac{1}{12}$. The linear approximation formula is

$$f(x) \approx f(a) + f'(a)(x-a)$$

So

$$f(8.1) \approx f(8) + f'(8)(8.1-8) = 2 + \frac{1}{12} \cdot 0.1 = 2 + \frac{1}{120}$$

33. Let $f(x) = \dfrac{1}{1+x^2}$.

(a) Find the local linearization of $f$ near $x = 1$.
**Answer:** $f(1) = \frac{1}{2}$ and $f'(1) = -\frac{1}{2}$, so the local linearization is $L(x) = \frac{1}{2} - \frac{1}{2}(x-1)$.

(b) Use the local linearization you found to estimate $f(1.1)$.
**Answer:** $f(1.1) \approx L(1.1) = 0.45$.

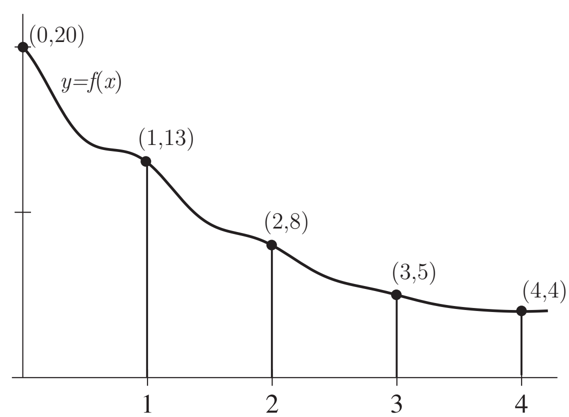
34. Let $C$ be the curve defined by the equation $xy + y^2 = 4$.

(a) Verify that the point $(3, 1)$ lies on the curve $C$.
**Answer:** $3 \cdot 1 + 1^2 = 4$.

(b) Find an equation for the tangent line to $C$ at the point $(3, 1)$.
**Answer:** $\dfrac{dy}{dx} = -\dfrac{y}{x+2y}$, so at $(3, 1)$ the slope is $m = -\frac{1}{5}$. The tangent line is $y - 1 = -\frac{1}{5}(x-3)$.

(c) If the curve defines a function $f(x)$ near $x = 3$, then estimate $f(3.2)$.
**Answer:** $f(3.2) \approx 1 + -\frac{1}{5}(3.2-3) = 0.96$

35. Consider the following function $f$.




(a) Find an approximation to the area under the graph of $f$, from $x = 0$ to $x = 4$, using a left endpoint Riemann sum with four rectangles.

**Answer:** $\displaystyle\int_0^4 f(x)\,dx \approx 1 \cdot (20 + 13 + 8 + 5) = 46$.

(b) Is your answer an overestimate of the actual area under the graph, or is it an underestimate? Please explain your reasoning.

**Answer:** It is an overestimate. Every one of the rectangles we used has larger area than is actually under the curve. (For decreasing functions, left sums always overestimate.

36. Calculate $\int_1^5 [3f(x)+4g(x)]\,dx$, given that

$$\int_1^2 f(x)\,dx = -1, \qquad \int_2^5 f(x)\,dx = 1, \qquad \int_{-1}^5 g(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^1 g(x)\,dx = 3.$$

answer

$$\begin{aligned}\int_1^5 \big[3f(x)+4g(x)\big]\,dx &= 3\int_1^5 f(x)\,dx + 4\int_1^5 g(x)\,dx \\ &= 3\left(\int_1^2 f(x)\,dx + \int_2^5 f(x)\,dx\right) + 4\left(\int_{-1}^5 g(x)\,dx - \int_{-1}^1 g(x)\,dx\right) \\ &= 3\cdot(-1+1)+4(-2-3) \\ &= -20.\end{aligned}$$

37. Consider the function $f(x) = x + \dfrac{1}{x}$ for $x \neq 0$.

(a) Find the critical points of $f$, and determine which are local minima of $f$, local maxima of $f$, or neither.

**Answer:** $f'(x) = 1 - 1/x^2$, so the critical points are when $x = 1$ or $x = -1$. ($x = 0$ would be a critical point if $f$ were defined there.) The point $x = 1$ is a local minimum since $f'(\frac{1}{2}) = -3 < 0$ and $f'(2) = \frac{3}{4} > 0$. Similarly $x = -1$ is a local maximum.

(b) Find the global maxima and minima of $f$.

**Answer:** As $x$ grows large, $f(x)$ looks approximately like $x$ (since the term $1/x$ becomes very small). So $f(x)$ can get arbitrarily large and small, and hence it has no global maxima or minima.

(c) Over what intervals is $f$ concave up? concave down?

**Answer:** $f''(x) = 2/x^3$, so $f$ is concave up when $x > 0$ and concave down when $x < 0$.

(d) Where are the inflection points of $f$?

**Answer:** $f$ has no inflection points since it is undefined at $x = 0$, where the concavity changes.

38. Let $F(x) = \int_0^{3x} f(t)\,dt$, where $f(t) = 2^{(t^2)}$.

(a) Calculate $F(0)$.

**Answer:** $F(0) = 0$.

(b) Using $n = 3$ subintervals and left-hand endpoints, estimate the value of $F(1)$. Begin by calculating $\Delta t$ and filling in the table of values below.

**Answer:** Plugging in $x = 1$ means we are trying to evaluate

$$\int_0^3 2^{t^2}\,dt$$

with $n = 3$ steps. Hence our $\Delta t = \frac{3-0}{3} = 1$. Filling in the table below and using the left-hand endpoints, we get the approximation

$$F(1) \approx 1 \cdot (1 + 2 + 16) = 19.$$

| $t$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(t)$ | 1 | 2 | 16 | 512 |

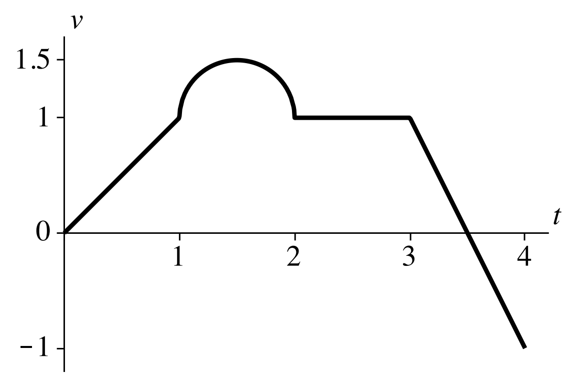
$\Delta t =$ 1

(c) Calculate $F'(x)$ and find the local linearization of $F(x)$ about $x = 0$.
**Answer:** $F'(x) = 3 \cdot 2^{9x^2}$, so that $F'(0) = 3$. Hence the local linearization is $L(x) = F(0) + F'(0)(x - 0) = 3x$.

(d) Using part (c), estimate the value $F(1)$ (this may not be a very good estimation).
**Answer:** The local linearization yields $F(1) \approx 3$.

39.

A particle's position at time $t$ is given by $s(t)$, and its velocity at time $t$ is given by $v(t)$. Given the graph of $v(t)$ below, if the particle has position $s = -1$ at $t = 1$, fill in the table of values for the position using *exact* values (i.e., without estimating). The graph below is made up of triangular, semicircular, and rectangular sections.
**Answer:** Geometrically we see that $s(1)-s(0)$ is the area of a triangle with base one and height one, so $s(1) - s(0) = \frac{1}{2}$. Now $s(2) - s(1)$ is the area of a square of side one plus half the area of a circle of radius $\frac{1}{2}$, so that $s(2) - s(1) = 1 + \frac{\pi}{8}$. Obviously $s(3) - s(2) = 1$ and $s(4) - s(3) = 0$.




| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $s(t)$ | $-\frac{3}{2}$ | $-1$ | $\frac{\pi}{8}$ | $1+\frac{\pi}{8}$ | $1+\frac{\pi}{8}$ |

40. The volume of water remaining in a leaking tank is given by the function $V = 50e^{-0.1t}$, where $V$ is measured in gallons and $t$ in minutes.

(a) How fast is the tank leaking after 10 minutes? Include appropriate units. **Answer:** $V'(10) = -5e^{-1}$ gallons per minute

(b) How much water remains in the tank after 10 minutes? Include appropriate units.
**Answer:** $V(10) = 50e^{-1}$ gallons

41. The rate at which water is leaking from an initially full 100-gallon tank is given by the function $r = -7e^{-0.1t}$, where $r$ is measured in gallons per minute and $t$ is measured in minutes. (The negative sign indicates a decrease as opposed to an increase in volume.)

(a) How fast is the tank leaking after 10 minutes? Include appropriate units. **Answer:** $r(10) = -7e^{-1}$ gallons per minute

(b) How much water remains in the tank after 10 minutes? Include appropriate units. **Answer:** $V(10) = V(0) + \int_0^{10} r(t)\,dt = 100 - 7\int_0^{10} e^{-0.1t}\,dt = 100 + 70(e^{-1} - 1) = 30 + 70e^{-1}$gallons

42. Let $F(x) = \displaystyle\int_2^x \frac{1}{\ln t}\,dt$. Find $F'(x)$. Is $F$ increasing or decreasing? What can you say about the concavity of $F$? **Answer:** $F'(x) = \frac{1}{\ln x}$. For $x > 1$, $F'(x) > 0$ so $F$ is increasing. It is not possible to have $x < 1$ since $\frac{1}{\ln t}$ is undefined at $t = 1$ (and we can't integrate across a discontinuity like that). Hence $F$ must always be increasing. Since $F''(x) = -\frac{1}{x(\ln x)^2}$, we know $F$ is always concave down.

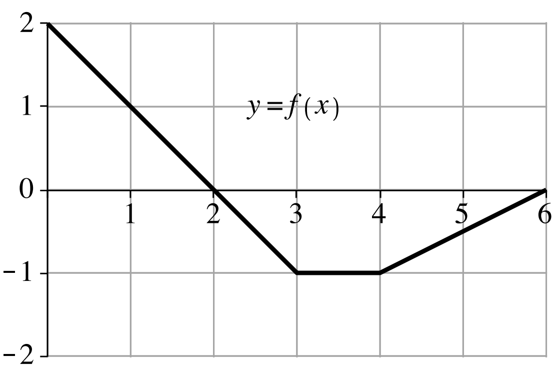
43.

Let $f(x)$ be the function whose graph is given to the right. Define

$$F(x) = \int_0^x f(t)\,dt.$$

Fill in the entries in the table that follows.

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $F(x)$ | $\frac{3}{2}$ | 2 | $\frac{3}{2}$ | $\frac{1}{2}$ | $-\frac{1}{4}$ |
| $F'(x)$ | 1 | 0 | -1 | -1 | $-\frac{1}{2}$ |
| $F''(x)$ | -1 | -1 | d.n.e. | d.n.e. | $\frac{1}{2}$ |




44. Let $f(x)$ be the function from the previous problem and $g(x)$ be given by the following table:

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $g(x)$ | 2 | -1 | 5 |
| $g'(x)$ | 3 | 4 | -2 |

If $h(x) = g(f(x))$, what is $h'(2)$? If $k(x) = f(g(x))$, what is $k'(2)$?
**Answer:** $h'(2) = g'(f(2))f'(2) = g'(0)f'(2) = 3 \cdot -1 = -3$.
$k'(2) = f'(g(2))g'(2) = f'(5)g'(2) = \frac{1}{2} \cdot -2 = -1$

45.

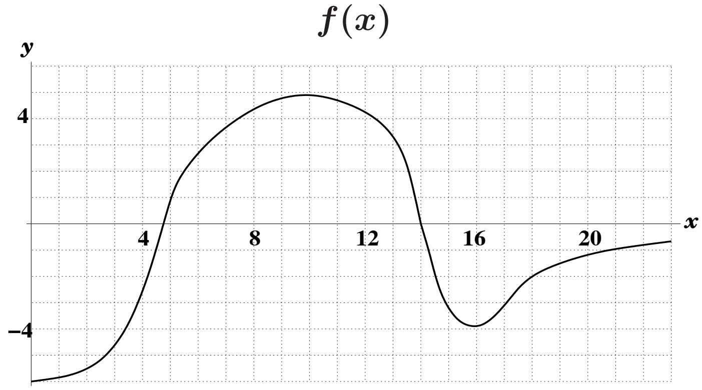
Let $f$ be the function whose graph is given below, and define a new function $F$ by the equation $F(x) = \int_0^x f(t)\,dt$.

Given below are several lists of numbers. Rank each list in order from smallest to largest.

(a) $0$, $f'(3)$, $f'(9)$, $f'(14)$, $f'(15)$
**Answer:** $f'(14) < f'(15) < 0 < f'(9) < f'(3)$.

(b) $0$, $F(3)$, $F(4)$, $F(6)$, $F(13)$, $F(14)$
**Answer:** $F(4) < F(6) < F(3) < 0 < F(13) < F(14)$.

(c) $0$, $F'(0)$, $F'(4)$, $F'(7)$, $F'(10)$, $F'(15)$, $F'(18)$, $F'(23)$
**Answer:** $F'0) < F'(15) < F'(4) < F'(18) < F'(23) < 0 < F'(7) < F'(10)$.




46. Find the slope of the tangent line to each of the following functions at $x = 2$.

(a) $f(x) = \sin(\cos x)$
**Answer:** $f'(2) = -\cos(\cos 2) \cdot \sin 2$

(b) $F(x) = \int_0^x e^{-t^2}\,dt$
**Answer:** $F'(2) = e^{-4}$

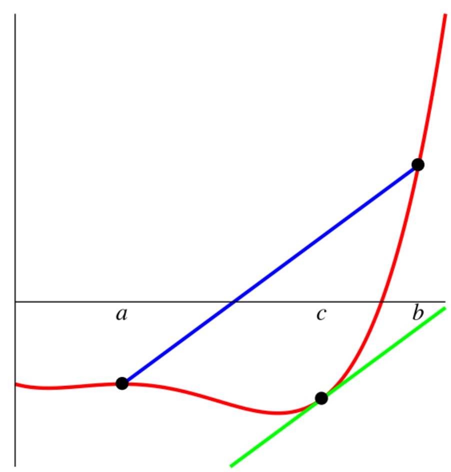
(c) $f(x) = x^x$
**Answer:** Write $f(x) = e^{\ln(x^x)} = e^{x \ln x}$ to get $f'(x) = e^{x\ln x}(1 + \ln x) = x^x(1 + \ln x)$. Thus $f'(2) = 4(1 + \ln 2)$.

47. State the Mean Value Theorem, and draw a picture to demonstrate the theorem visually.

**Answer:**
If $f$ is differentiable on $[a, b]$, then there is a point $c$ such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

On the graph it says that given any secant line (shown in blue) there is some tangent line in between (shown in green) which is parallel.



48. Let $f(x) = (x-1)^2 e^{-x}$.

(a) State the domain of $f(x)$.
**Answer:** All reals.

(b) What is the behavior of $f(x)$ as $x \to +\infty$ and $x \to -\infty$?
**Answer:** As $x \to +\infty, f(x) \to 0$. As $x \to -\infty, f(x) \to +\infty$.

(c) Find the intervals of increase and decrease of $f(x)$, and all local maxima and minima of $f(x)$.
**Answer:** $f'(x) = -(x-1)^2 e^{-x} + 2(x-1)e^{-x} = (x-1)(3-x)e^{-x}$. So $f(x)$ is decreasing for $x < 1$, increasing for $1 < x < 3$, and decreasing again for $x > 3$. It has a local maximum at $x = 3$ and a local minimum at $x = 1$.

(d) Find the inflection points, and the intervals where the function is concave up/down.
**Answer:** $f''(x) = -(x-1)(3-x)e^{-x} + (3-x-x+1)e^{-x} = e^{-x}(x^2 - 6x + 7)$. This doesn't factor nicely, but the quadratic formula gives its roots as $x = 3 \pm \sqrt{2}$. These are both inflection points, since we can test $f''(3) < 0, f''(5) > 0$, and $f''(1) > 0$. Hence the function is concave up for $x < 3 - \sqrt{2}$, concave down for $3 - \sqrt{2} < x < 3 + \sqrt{2}$, and concave up for $x > 3 + \sqrt{2}$.

49. The acceleration due to gravity on the moon is $1.6\frac{m}{s^2}$. An object is thrown upwards from a height of 2.4 m, with an initial velocity of $1.6\frac{m}{s}$. **Answer:** First, acceleration is $a(t) = -1.6$. Anti-differentiate to get velocity: $v(t) = -1.6t + C$. When $t = 0$, velocity is $+1.6$ (the object is throw upwards). Substituting: $1.6 = 0 +$ C, so $C = 1.6$, and $v(t) = -1.6t + 1.6$. Antidifferentiate again to get position above ground: $s(t) = -.8t^2 + 1.6t + C_2$. When $t = 0$, $s(0) = 2.4$. Subsituting gives $2.4 = 0 + 0 + C_2$, so $s(t) = -.8t^2 + 1.6t + 2.4$.

(a) What height does the object reach? **Answer:** The object hits its maximum height when $v(t) = 0$. Solving $0 = -1.6t + 1.6$ gives a times of $t = 1$ seconds. Substituting this into the position function gives $s(1) = -.8 + 1.6 + 2.4 = 3.2$ meters.

(b) How fast is the object traveling when it lands? **Answer:** This time we need to start by finding the time when the object hits ground, so set $s(t) = 0$ and solve for $t$. $0 = -.8t^2 + 1.6t + 2.4 = -.8(t^2 - 2t - 3) = -.8(t-3)(t+1)$. We take the positive solution for $t$, $t = 3$. Substituting this into elocity gives $v(3) = -1.6 \cdot 3 + 1.6 = -3.2$. The object hits ground with a downward velocity of $-3.2$ meters per second.

**More practice from the last two chapters:**

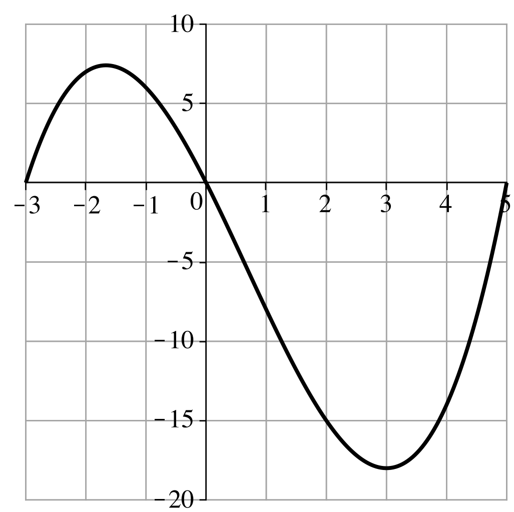
50. A village wishes to measure the quantity of water that is piped to a factory during a typical morning. A gauge on the water line gives the flow rate (in cubic meters per hour) at any instant. The flow rate is about 100 $m^3$/hr at 6 am and increases steadily to about 280 $m^3$/hr at 9 am.

(a) Using only this information, give your best estimate of the total volume of water used by the factory between 6 am and 9 am.
**Answer:** If we assume the flow rate is increasing linearly, then we just need to find the area under the trapezoid. We get Volume$= \frac{1}{2} \cdot 3 \cdot (100 + 280) = 570$ $m^3$.

(b) How often should the flow rate gauge be read to obtain an estimate of this volume to within 6 $m^3$?
**Answer:** Now we are worried about the flow rate possibly not being a straight line, but rather some other increasing function. So we would need to use essentially left-hand sums and right-hand sums, and to make them accurate we need the difference between the left and right sums to be less than or equal to 6. Since $R_n - L_n = \frac{(280-100)\cdot(9-6)}{n} = \frac{540}{n}$, we can make this less than or equal to 6 by making $n \geq 540/6 = 90$. So we should be reading the gauge 90 times in 3 hours, or once every two minutes.

51.
Suppose a function $f(x)$ is graphed as below.

(a) Using just the figure, estimate $\int_{-3}^{5} f(x)\,dx$.
**Answer:** The left region can be approximated by a triangle of height 8 and base 3 to get area 12. The right region can be approximated with a triangle of height 20 and base 5 to get area 50. So the integral is about 12-50=-38.




(b) If you knew the function were given by the formula $f(x) = \frac{1}{2}x(x+3)(x-5)$, what would the integral be? How far off is your graphical estimate?
**Answer:** The integral is $\int_{-3}^{5} \frac{1}{2}(x^3 - 2x^2 - 15x)\,dx = \frac{1}{2}(\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{15}{2}x^2)\Big|_{x=-3}^{x=5} = -\frac{128}{3} = -42.67$. Off by about five units. Oh well.

52. Find the area of the region between the line $y = 1$ and one arch of $y = \sin\theta$.
**Answer:** One arch occurs between $\theta = 0$ and $\theta = \pi$. So the area is

$$\int_0^\pi (1 - \sin\theta)\,d\theta = (\theta + \cos\theta)\Big|_{\theta=0}^{\theta=\pi} = (\pi - 1) - (0 + 1) = \pi - 2.$$

53. Find the area of the region between the parabola $y = 4 - x^2$ and the $x$-axis.
**Answer:** They cross at $x = \pm 2$, so the area is

$$\int_{-2}^{2} (4 - x^2)\,dx = 2\int_0^2 (4 - x^2)\,dx = 2(4x - \frac{1}{3}x^3)\Big|_{x=0}^{x=2} = \frac{32}{3}.$$

54. If $\int_2^5 \big(2f(x) + 3\big)\,dx = 17$, find $\int_2^5 f(x)\,dx$.
**Answer:** Let $A = \int_2^5 f(x)\,dx$. Then $2A + \int_2^5 3\,dx = 17$, so $2A + 9 = 17$, and thus $A = 4$.

55. Find $\int_{-1}^{1} |x|\,dx$ geometrically.
**Answer:** It's the area of two triangles, each having height 1 and base 1. So the integral is 1.

56. An old rowboat has sprung a leak. Water is flowing from the boat at a rate, $r(t)$, given in the following table.

| $t$ minutes | 0 | 5 | 10 | 15 |
|---|---|---|---|---|
| $r(t)$ liters/min | 12 | 20 | 24 | 16 |

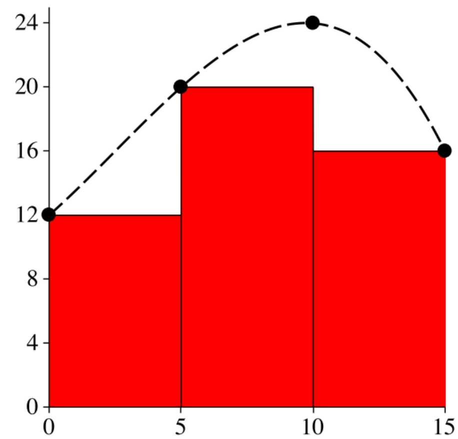
(a) Compute upper and lower estimates for the volume of water that has flowed into the boat during the 15 minutes.
**Answer:** To get the upper estimate, we use the maximum value over each interval, which is

$$\text{Max Volume} = 5 \cdot (20 + 24 + 24) = 340 \text{ liters}.$$

To get the lower estimate, we use the minimum value over each interval, which is

$$\text{Min Volume} = 5 \cdot (12 + 20 + 16) = 240 \text{ liters}.$$

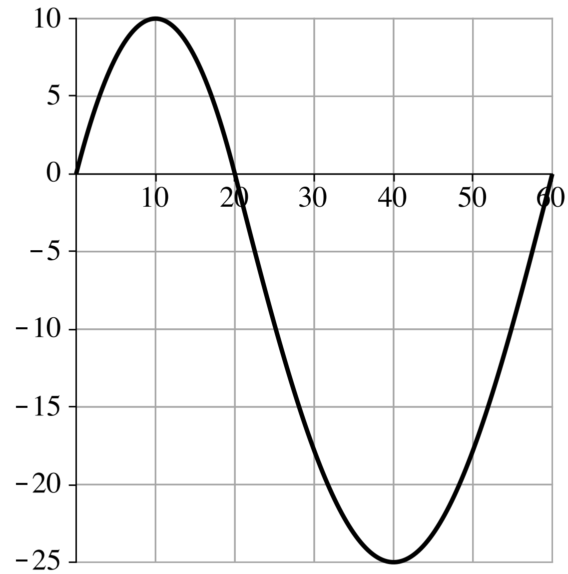
(b) Draw a graph to illustrate the lower estimate.
**Answer:** The lower sum is represented by the area of the red rectangles. The data points are represented as dots, and we've connected the dots with a possible curve.




57. A bicyclist is pedaling along a straight road for one hour with a velocity $v$ shown in the figure. She starts out five kilometers from the lake and positive velocities take her toward the lake. Note that the horizontal units are in minutes while the vertical units are in kilometers per hour.

(a) Does the cyclist ever turn around? If so, at what time(s)?
**Answer:** The cyclist turns around at $t = 20$ minutes. It seems like the cyclist is also about to turn around at $t = 60$ minutes, if the velocity graph continues above the $x$-axis.

(b) When is she going the fastest? How fast is she going then? Toward the lake or away?
**Answer:** Her maximum speed is at $t = 40$ minutes, when she is going -25 kilometers per hour (away from the lake).




(c) When is she closest to the lake? Approximately how close to the lake does she get?
**Answer:** The closest she gets is when $t = 20$ minutes; she has been riding towards it until then, and she turns around and starts riding away. The area under the curve from 0 to 20 is a little more than the area of the triangle $\frac{1}{2} \cdot \frac{1}{3} \cdot 10 = 1.66$ but definitely less than the area of the rectangle $10 \cdot \frac{1}{3} = 3.33$, so let's say it's about 2 kilometers. Then she gets about 3 kilometers away.

(d) When is she farthest from the lake? Approximately how far from the lake is she then?
**Answer:** She is farthest from the lake when $t = 60$ minutes. At this time she is roughly $5 - 2 + 9 = 12$ kilometers away. (I approximated the small hump area by 2 and the large hump area by 9).

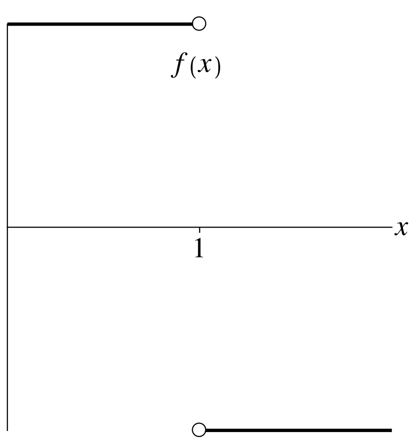
58. The area under $1/\sqrt{x}$ on the interval $1 \leq x \leq b$ is equal to 6. Find the value of $b$ using the Fundamental Theorem. **Answer:** $\int_1^b x^{-1/2}dx = 2\sqrt{b} - 2 = 6$. So $b = 16$.

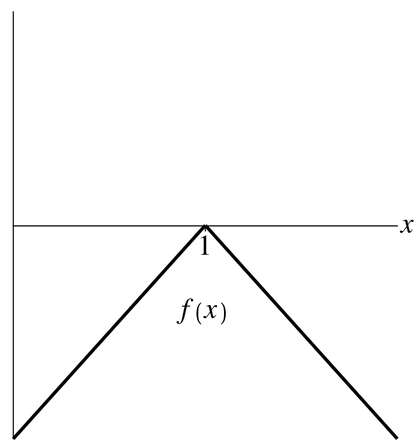
59. Find the exact area of the region bounded by the $x$-axis and the graph of $y = x^3 - x$. **Answer:** The cubic crosses the axis at three points: $x = 0$, $x = 1$, and $x = -1$. There is both a region about the axis (for $-1 < x < 0$) and a region below the axis (for $0 < x < 1$), and we want them both. By symmetry they will have the same area. So if we just compute one, we'll know both.

$$\int_0^1 (x^3 - x)dx = \left(\frac{1}{4}x^4 - \frac{1}{2}x^2\right)\Big|_{x=0}^{x=1} = -\frac{1}{4},$$

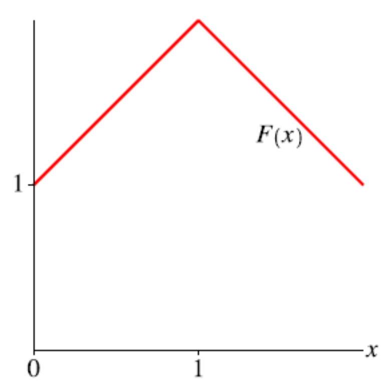
which is negative as we expect since this region is below the axis. Therefore the area of each region is $\frac{1}{4}$, and the total area is $\frac{1}{2}$.

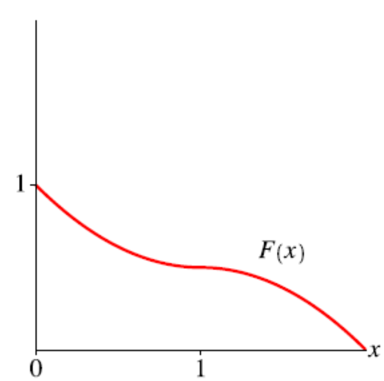
60. For each of the graphs $f(x)$ shown below, sketch a graph of $F(x)$ such that $F'(x) = f(x)$ and $F(0) = 1$.




**Answer:**



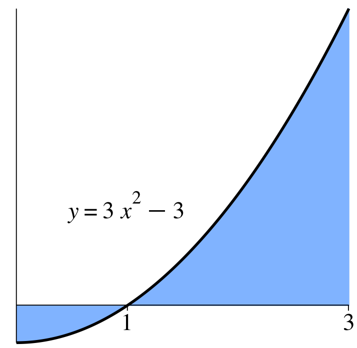
61.

Find the area of the shaded region in the figure between $y = 3x^2 - 3$ and the $x$-axis.

**Answer:** Set up two separate integrals, since which function is higher than the other changes at $x = 1$. So the area is:

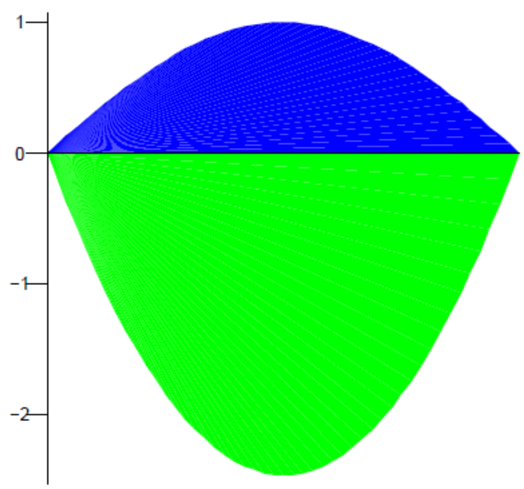
$$\int_0^1 [0 - (3x^2 - 3)]dx + \int_1^3 [(3x^2 - 3) - 0]dx$$

$$= (3x - x^3)\Big|_{x=0}^{x=1} + (x^3 - 3x)\Big|_{x=1}^{x=3} = 22.$$




62. Sketch the parabola $y = x(x - \pi)$ and the curve $y = \sin x$, showing their points of intersection. Find the exact area between the two graphs.

**Answer:** Here is a picture. The top one is the sine curve, and the bottom one is the parabola. They intersect at $x = 0$ and $x = \pi$.

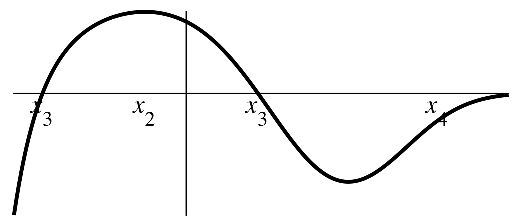



The area is

Area $= \int_0^\pi (\sin x - x^2 + \pi x)dx = (-\cos x - \frac{1}{3}x^3 + \frac{1}{2}\pi x^2)\Big|_{x=0}^{x=\pi} = (1 - \frac{1}{3}\pi^3 + \frac{1}{2}\pi^3) - (-1 - 0 + 0) = 2 + \frac{\pi^3}{6}$.

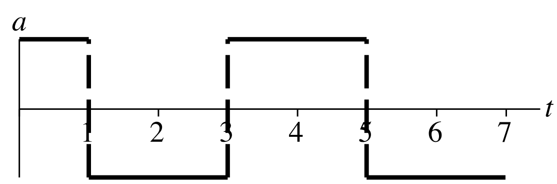
63.

The graph of $f'(x)$ is given. Sketch a possible graph for $f(x)$. Mark the points $x_1, \ldots, x_4$ on your graph and label local maxima, local minima, and inflection points on your graph.

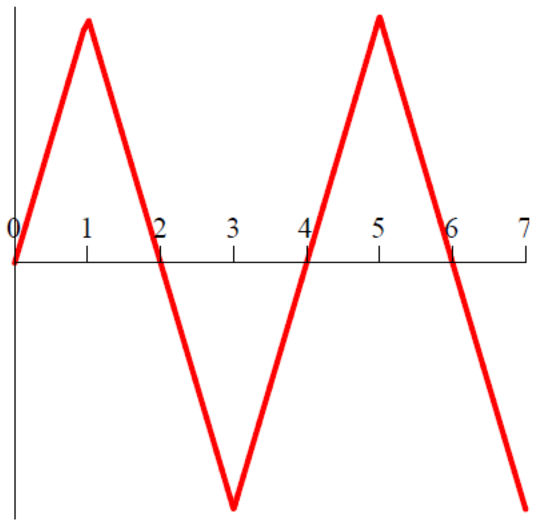


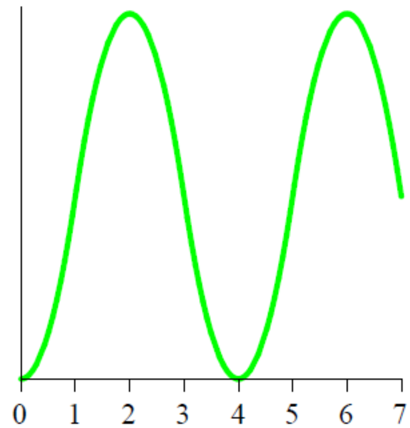

**Answer:** Note the typo, the first point should have been labelled $x_1$. $f(x)$ has a local minimum at $x_1$ (the derivative switches from negative to positive there). $f(x)$ has an inflection point at $x_2$ (the second derivative switches from positive to negative there), $f(x)$ has a local maximum at $x_3$ (the derivative switches from positive to negative there). At $x_4$, the derivative has an inflection point, but that doesn't tell us anything particular about $f(x)$. Your graph should approach $+\infty$ on the left and some knd of horizontal asymptote on the right.

64. The acceleration, $a$, of a particle as a function of time is shown in the figure. Sketch graphs of velocity and position against time. The particle starts at rest at the origin.



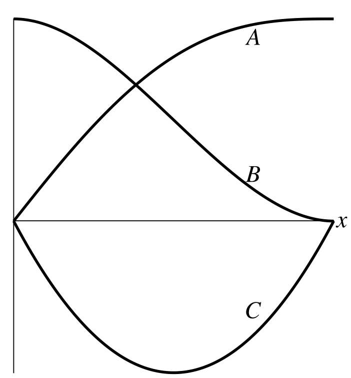

**Answer:** The velocity must be piecewise linear since accleration is piecewise constant. Also, physically velocity is continuous even if acceleration is not. Since it starts at rest, $v(0) = 0$. So $v(1) = 1$, $v(3) = -1$, etc. Then we integrate again to get position, which is piecewise parabolic (and differentiable everywhere, even though velocity is not). Velocity is graphed in red, and position is in green.




65.

The graphs of three functions are given in the figure. Determine which is $f$, which is $f'$, and which is $\int_0^x f(t)\,dt$. Explain your answer.




**Answer:** Let's look for interesting features like turning points and inflections points. Graph C is the only one with a turning point (and it occurs in the middle), and if we had the graph of the derivative of C, then that derivative graph would have a zero in the middle. However, none of the graphs have zeros in the middle. That means C must be the highest derivative, which is $f'$.

Now when $f'$ has a turning point (in the middle), the function $f$ must have an inflection point. Graph B has an inflection point in the middle (at the correct location), so graph B must be $f$.

Graph A is then $\int_0^x f(t)dt$ by elimination. The first check is that graph A goes through the origin, as it should. We should also be checking then that the derivative of graph A is $f$ (second part of the Fundamental Theorem of Calculus). Clues for this are that A is strictly increasing (because $f$ is positive), and that A is concave down (because $f$ is decreasing.