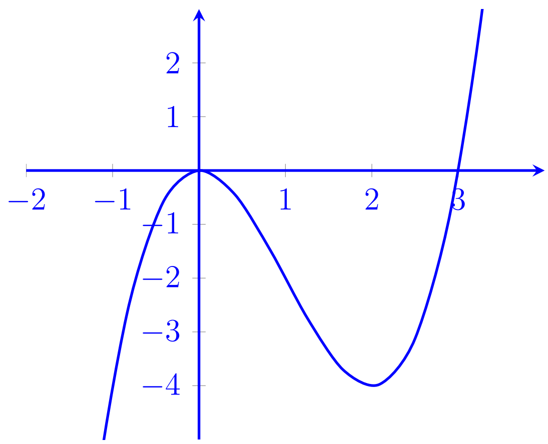
Review Sheet for Third Midterm
Mathematics 1300, Calculus 1

1. For $f(x) = x^3 - 3x^2$ on $-1 \leq x \leq 3$, find the critical points of f, the inflection points, the values of f at all these points and the endpoints, and the absolute maxima and minima of f. Then sketch the graph, indicating clearly where f is increasing or decreasing and its concavity.

   Answer: $f'(x) = 3x^2 - 6x$. Set equal to 0 and solve to find critical numbers at $x = 0$ and $x = 2$. Using first derivative test, we find the derivative switches from + to - at $x = 0$, so $f(x)$ has a local maximum of $y = 0$ there. The derivative switches from - to + at $x = 2$ so $f(x)$ has a local minimum of $y = -4$ there. $f''(x) = 6x - 6$, the second derivative is zero at $x = 1$. We find that the second derivative switches from - to + at $x = 1$, so $f(x)$ has an inflection point at $(1, -2)$.

   Now for the absolute extrema on $[-1, 1]$. Note that the critical number $x = 2$ is not in the interval, so we just need to substitute the endpoints and $x = 0$. $f(-1) = -4$, $f(0) = 0$ and $f(1) = -2$. The absolute maximum on the interval is 0, occurring at $x = 0$ and the absolute minimum is -4, occurring at $x = -1$.

2. For $f(x) = x + \sin x$ on $0 \leq x \leq 2\pi$, find the critical points of f, the inflection points, the values of f at all these points and the endpoints, and the absolute maxima and minima of f. Then sketch the graph, indicating clearly where f is increasing or decreasing and its concavity.

   Answer: The only critical point is $x = \pi$, which is also the only inflection point. Since $f(0) = 0$, $f(\pi) = \pi$, and $f(2\pi) = 2\pi$, the absolute maximum is 2π while the absolute minimum is 0. The function is always increasing and is concave down on $[0, \pi]$ and concave up on $[\pi, 2\pi]$.

3. For $f(x) = \dfrac{4x^2}{x^2 + 1}$, find the critical points of f, the inflection points, the values of f at all these points, the limits as $x \to \pm\infty$, and the absolute maxima and minima of f. Then sketch the graph, indicating clearly where f is increasing or decreasing and its concavity.

Answer: The only critical point is $x = 0$, which is a local minimum. The inflection points are at $x = \pm 1/\sqrt{3}$. We have $f(0) = 0$ and $f(\pm 1/\sqrt{3}) = 1$. As $x \to \pm\infty$, $f(x) \to 4$.

4. Find the exact absolute maximum and minimum values of the function

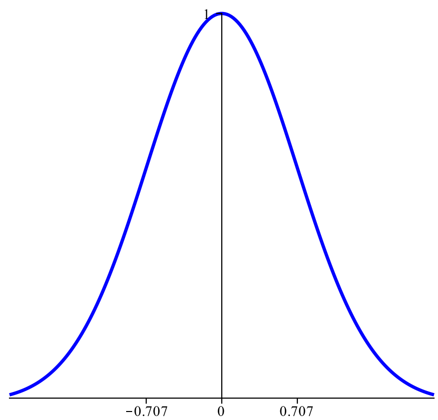
$$h(z) = \frac{1}{z} + 4z^2 \text{ for } z > 0$$

Answer: There is no absolute maximum. The absolute minimum occurs at $x = \frac{1}{2}$ and the absolute minimum value is 3.

5. Use derivatives to identify local maxima and minima and points of inflection. Then graph the function.

$$f(x) = e^{-x^2}.$$

Answer: The only local extremum is $x = 0$ which is a local maximum. The inflection points occur at $x = \pm\frac{1}{\sqrt{2}}$. Asymptotically as $x \to \pm\infty$ the function approaches zero.



6. (a) Find all critical points and all inflection points of the function $f(x) = x^4 - 2ax^2 + b$. Assume a and b are positive constants.

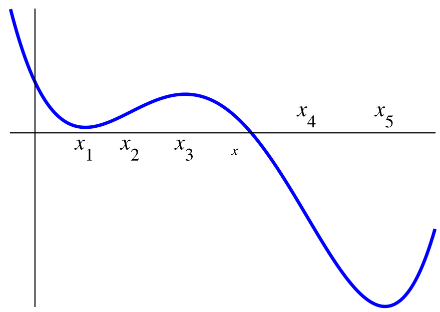
   Answer: Critical points occur at $x = 0$ and $x = \pm\sqrt{a}$. Inflection points occur at $x = \pm\sqrt{a/3}$.

   (b) Find values of the parameters a and b if f has a critical point at the point $(2, 5)$.

   (c) If there is a critical point at $(2, 5)$, where are the inflection points?

   Answer: If there is a critical point at $(2, 5)$, then $a = 4$ and $b = 21$, and the inflection points happen at $x = \pm\sqrt{4/3}$.

7. For the function, f, graphed in the Figure:

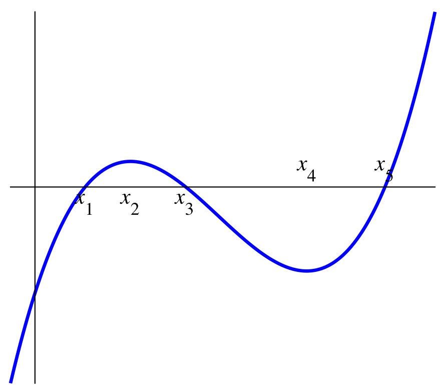


(a) Sketch $f'(x)$.

(b) Where does $f'(x)$ change its sign?

(c) Where does $f'(x)$ have local maxima or minima?

Answer: Here is the derivative:



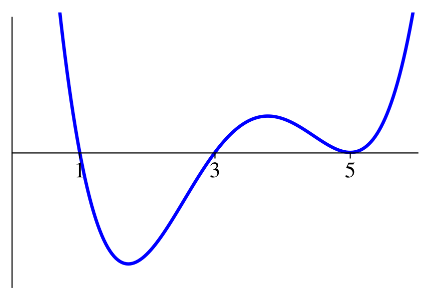
$f'(x)$ changes sign at x_1, x_3, and x_5. It has a local maximum at x_2 and a local minimum at x_4.

8. Using your answer to the previous problem as a guide, write a short paragraph (using complete sentences) which describes the relationships between the following features of a function f:

- The local maxima and minima of f.
- The points at which the graph of f changes concavity.
- The sign changes of f'.
- The local maxima and minima of f'.

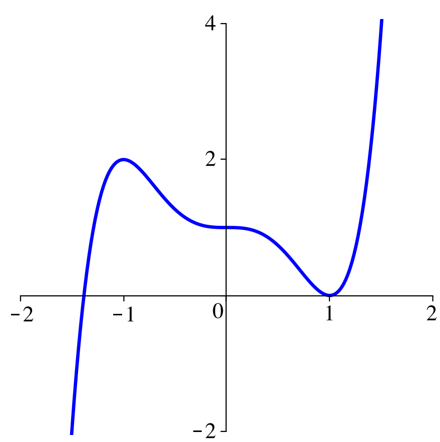
Answer: You answer should include the points that the local maxima and minima of f are exactly the sign changes of f'. A local minimum occurs where $f'(x)$ switches from - to +, while a local maximum occurs where $f'(x)$ switches from + to -. The points at which the graph of f changes concavity are exactly the local maxima and local minima of f', because this is where $f''(x)$ switches sign.

9. The figure shows a graph of $y = f'(x)$ (not the function f). For what values of x does f have a local maximum? A local minimum?



Answer: f has a local maximum at $x = 1$ and a local minimum at $x = 3$. The point at $x = 5$ is a stationary point which is neither a local maximum nor a local minimum.

10. On the graph of f' shown in the figure, indicate the x-values that are critical points of the function f itself. Are they local minima, local maxima, or neither?

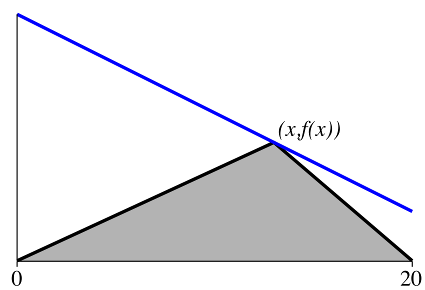


Answer: The critical points of f occur at $x = -1.5$ and $x = 1$. The point at $x = -1.5$ is a local minimum, and the point at $x = 1$ is neither.

11. Find a formula for the function described: a cubic polynomial with a local maximum at $x = 1$, a local minimum at $x = 3$, a y-intercept of 5, and an x^3 term whose coefficient is 1.

Answer: It must be of the form $y = x^3 + ax^2 + bx + c$ for some constants a, b, and c. The information given tells us $c = 5$, $3 + 2a + b = 0$, and $27 + 6a + b = 0$, so that $a = -6$ and $b = 9$.

12. Find the x-value maximizing the shaded area. One vertex is on the graph of $f(x) = x^2/3 - 50x + 1000$ (shown in blue).



Answer: The area of the region is $A = 10(x^2/3 - 50x + 1000)$. The critical point is at $x = 75$, which is not in the domain $0 \leq x \leq 20$. At the endpoint $x = 0$ we have $A = 10,000$, and at the endpoint $x = 20$ we have $A = \frac{4,000}{3}$. So the maximum is at $x = 0$.

13. Given that the surface area of a closed cylinder of radius r cm and height h cm is 8 cm^2, find the dimensions giving the maximum volume.

Answer: The surface area is $S = 2\pi rh + 2\pi r^2 = 8$, so that $h = \frac{4}{\pi r} - r$. The volume is

$$V = \pi r^2 h = 4r - \pi r^3$$

which has a critical point at $r = \frac{2}{\sqrt{3\pi}}$. The endpoints are $r = 0$ (which gives zero volume) and $h = 0$ (which also gives no volume), so this critical point must be the maximizer. At this critical point we have $h = \frac{4}{\sqrt{3\pi}}$.

14. Do the last two problems from Project 11.

Answer: Solutions are posted on the "Projects" page of the website.

15. Find the range of the function $f(x) = x^3 - 6x^2 + 9x + 5$, if the domain is $0 \leq x \leq 5$.

Answer: The range is $[5, 25]$, determined by finding the absolute extrema on the domain, and the fact that $f(x)$ is a continuous function.

16. A landscape architect plans to enclose a 3000 square foot rectangular region in a botanical garden. She will use shrubs costing \$45 per foot along three sides and fencing costing \$20 per foot along the fourth side. Find the minimum total cost.

Answer: The area constraint is $xy = 3000$, and the fencing cost is $C = 90x + 65y$. The optimum values are $x = 46.5$ and $y = 64.5$ feet.

17. A piece of wire of length L cm is cut into two pieces. One piece, of length x cm, is made into a circle; the rest is made into a square. Find the value of x that makes the sum of the areas of the circle and square a minimum. Find the value of x giving a maximum.

Answer: Let x be the amount of wire used for the circle. Then the perimeter of the square is $L - x$, so the side length of the square is $y = \frac{L-x}{4}$. The radius of the circle is $r = \frac{x}{2\pi}$. The sum of the areas is

$$A = \pi r^2 + y^2 = \frac{x^2}{4\pi} + \frac{(L-x)^2}{16}.$$

The critical point is at $x = \frac{L\pi}{\pi+4} = 0.44L$.

18. Does L'Hopital's rule apply to the limit? If so, evaluate it.

$$\lim_{x\to 0} \frac{x}{e^x}.$$

Answer: No. The numerator is going to 0 but the denominator is going to 1. So the limit is 0 without using L'Hopital's rule.

19. Does L'Hopital's rule apply to the limit? If so, evaluate it.

$$\lim_{x\to 1} \frac{\sin \pi x}{x-1}.$$

Answer: Yes. Both the numerator and denominator are approaching zero.

$$\lim_{x\to 1} \frac{\sin \pi x}{x-1} = \lim_{x\to 1} \frac{\pi \cos \pi x}{1} = -\pi.$$

20. Determine whether the limit exists, and if possible evaluate it.

$$\lim_{t\to 0} \frac{e^t - 1 - t}{t^2}$$

Answer: The limit approaches $\frac{1}{2}$, by applying L'Hopital's rule twice.

21. Determine whether the limit exists, and if possible evaluate it.

$$\lim_{t\to 0^+} \frac{3\sin t - \sin 3t}{3\tan t - \tan 3t}$$

Answer: It is of the form $\frac{0}{0}$. So we use L'Hopital's rule to get

$$\begin{aligned}
\lim_{t\to 0^+} \frac{3\sin t - \sin 3t}{3\tan t - \tan 3t} &= \lim_{t\to 0^+} \frac{3\cos t - 3\cos 3t}{3\sec^2 t - 3\sec^2 3t} = \frac{0}{0} \\
&= \lim_{t\to 0^+} \frac{-3\sin t + 9\sin 3t}{6\sec^2 t\tan t - 18\sec^2 3t\tan 3t} = \frac{0}{0} \\
&= \lim_{t\to 0^+} \frac{-3\cos t + 27\cos 3t}{6\sec^4 t + 12\sec^2 t\tan^2 t - 54\sec^4 3t - 108\sec^2 3t\tan^2 3t} \\
&= \frac{24}{-48} = -\frac{1}{2}.
\end{aligned}$$

22. True or false? If true, explain how you know. If false, give a counterexample. If $f'(p) = 0$, then $f(x)$ has a local minimum or local maximum at $x = p$.

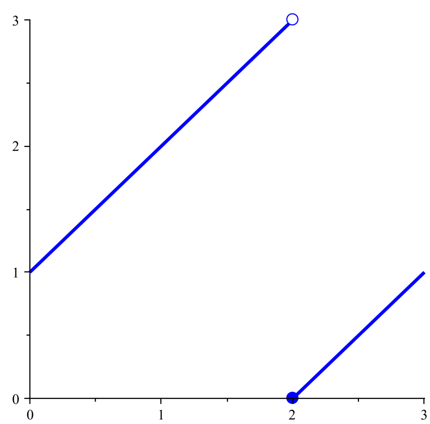
Answer: False. Let $f(x) = x^3$ and $p = 0$. Then $f'(0) = 0$, but $x = 0$ is neither a local minimum nor a local maximum.

23. True or false? If true, explain how you know. If false, give a counterexample. If $f''(p) = 0$, then the graph of f has an inflection point at $x = p$.

Answer: False. Let $f(x) = x^4$ and $p = 0$. Then $f''(0) = 0$, but $f''(x) = 12x^2$ which never changes sign.

24. Sketch the graph of a function defined at every point of $[0, 2]$ which has an absolute minimum but no absolute maximum. Why does your graph not contradict the Extreme Value Theorem?

Answer: Here is one.

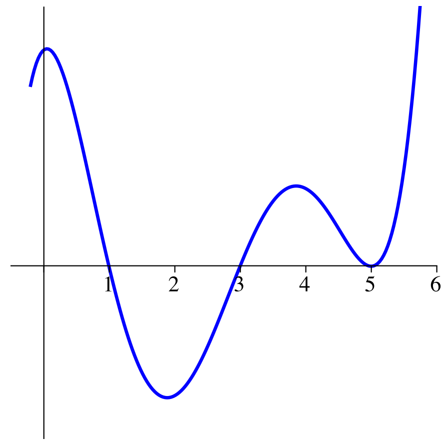


The global minimum happens at $x = 2$, but there is no global maximum. The Extreme Value Theorem only applies to *continuous* functions, and this one is not.

25. Precisely state the Mean Value Theorem.

Answer: See page 272 of the text.

26. The graph of $y = f''(x)$ is shown. Where are the inflection points of $y = f(x)$?



Answer: The inflection points are at $x = 1$ and $x = 3$, where the sign of $f''(x)$ changes. $x = 5$ is not an inflection point.

27. Find the absolute maximum and minimum of the function $f(x) = 9x^{1/3} - x^3$ on the closed interval $[-1, 8]$.

    Answer: The absolute maximum is 8, occurring at $x = 1$, and the absolute minimum is -494, occuring at $x = 8$.

28. Which point on the parabola $y = x^2 - x$ minimizes the square of the distance to the point $(1, 1)$?

    Answer: The square of the distance is

    $$D = (x - 1)^2 + (x^2 - x - 1)^2.$$

    Critical points happen at $x = 0$ and $x = \frac{3}{2}$. We compute $D(0) = 2$ and $D(\frac{3}{2}) = \frac{5}{16}$, so the minimizer is $(\frac{3}{2}, \frac{3}{4})$.

29. What is the largest area a rectangle with a perimeter of 40 inches can have?

    Answer: Set up $2x + 2y = 40$ and $A = xy$ to get $A = x(20 - x)$. The domain for x is $[0, 20]$, because x cannot be negative, and neither can y. $A'(x) = 20 - 2x$, which is zero when $x = 10$. When $x = 0$ or $x = 20$ the area is zero, so the maximum occurs at the critical point, when $x = 10$. The maximum area is $A = 100$.

30. You have 80 feet of fencing and want to enclose a rectangular area up against a long, straight wall (using the wall for one side of the enclosure and the fencing for the other three sides of the enclosure). What is the largest area you can enclose?

    Answer: Letting x be the length of the side perpendicular to the wall and y be the length of the side parallel to the wall, we have $2x + y = 80$ and $A = xy$ to get $A = x(80 - 2x)$. The domain is that x must be in the interval $[0, 40]$. Taking the derivative of $A(x)$ and setting it equal to 0 gives a critical point at $x = 20$. Evaluating the area at the endpoints and the critical point gives a maximum area of 800 square feet.

31. Compute

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n.$$

Answer: e^2.

32. Compute

$$\lim_{x\to 1}\frac{\ln x}{x^2-1}.$$

Answer: $\frac{1}{2}$.

33. Compute

$$\lim_{x\to 2}\frac{x^2-3x+2}{x+5}.$$

Answer: 0 .

34. Let f be a continuous function with domain $[-1,3]$, and assume that f is differentiable on $(-1,3)$. Which of the following statements about f are definitely true?

(a) If f has a critical point at $x=2$, then f has either a local maximum or a local minimum at $x=2$.

(b) If f has a critical point at $x=1$ and $f''(x)<0$, then f has a local maximum at $x=1$.

(c) If f has a critical point at $x=1$ and $f''(x)<0$, then f has an absolute maximum at $x=1$.

(d) If $f(3)>f(-1)$, then f has an absolute maximum at $x=3$.

(e) If $f'(0)=0$ and $f(0)>f(3)>f(-1)$, then f has an absolute maximum at $x=0$.

(f) If the only critical point of f occurs at $x=2$, then f must have an absolute maximum at either $x=-1,2$, or 3.

(g) It is possible that f might not have an absolute minimum on this domain.

(h) If f has a local minimum at $x=2$ then $f'(2)=0$.

(i) If $f''(x)>0$ for all values of x in the domain, then $f(3)\geq f(-1)$.

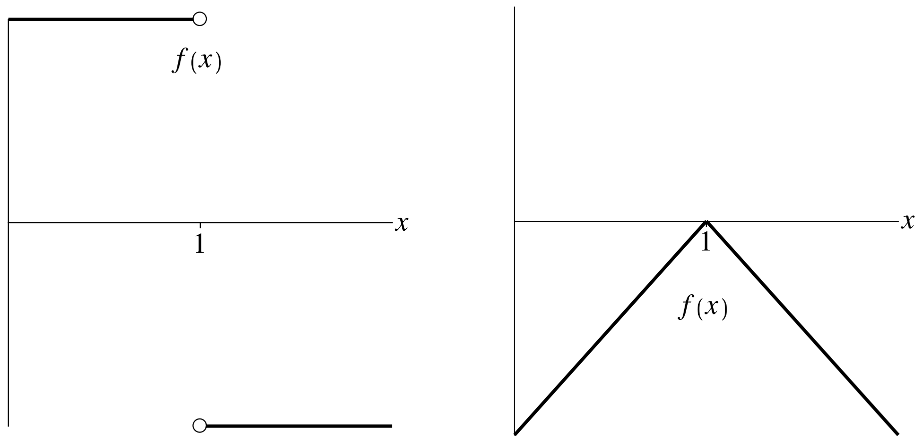
(j) If $f''(x)>0$ for all values of x in the domain, then $f'(3)\geq f'(-1)$.

(k) If f has an inflection point at $x=1$ then $f'(x)$ has either a local maximum or a local minimum at $x=1$.

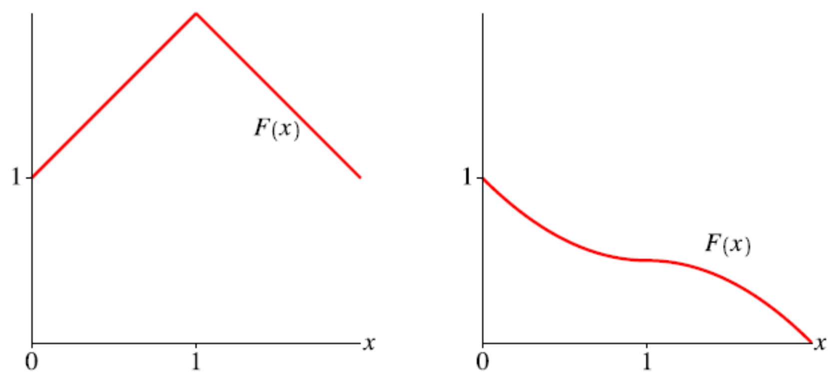
(l) If $f''(1)=0$ then f has an inflection point at $x=1$.

Answer: True: (b), (f), (h), (j), (k) False: (a), (c), (d), (e), (g), (i), (l)

35. For each of the graphs $f(x)$ shown below, sketch a graph of $F(x)$ such that $F'(x)=f(x)$ and $F(0)=1$.



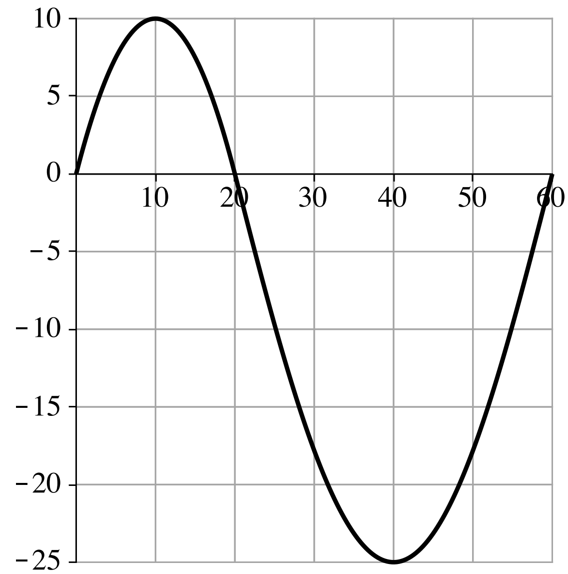
Answer:



36. A bicyclist is pedaling along a straight road for one hour with a velocity v shown in the figure. She starts out five kilometers from the lake and positive velocities take her toward the lake. Note that the horizontal units are in minutes while the vertical units are in kilometers per hour.

(a) Does the cyclist ever turn around? If so, at what time(s)?
Answer: The cyclist turns around at $t = 20$ minutes. It seems like the cyclist is also about to turn around at $t = 60$ minutes, if the velocity graph continues above the x-axis.

(b) When is she going the fastest? How fast is she going then? Toward the lake or away?
Answer: Her maximum speed is at $t =$ 40 minutes, when she is going -25 kilometers per hour (away from the lake).



(c) When is she closest to the lake? Approximately how close to the lake does she get?
Answer: The closest she gets is when $t = 20$ minutes; she has been riding towards it until then, and she turns around and starts riding away. The area under the curve from 0 to 20 is a little more than the area of the triangle $\frac{1}{2} \cdot \frac{1}{3} \cdot 10 = 1.66$ but definitely less than the area of the rectangle $10 \cdot \frac{1}{3} = 3.33$, so let's say it's about 2 kilometers. Then she gets about 3 kilometers away.

(d) When is she farthest from the lake? Approximately how far from the lake is she then?
Answer: She is farthest from the lake when $t = 60$ minutes. At this time she is roughly $5 - 2 + 9 = 12$ kilometers away. (I approximated the small hump area by 2 and the large hump area by 9).

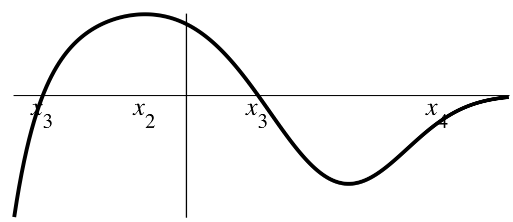
37. Suppose that $f(2) = 4$, and that the table below gives values of f' for x in the interval $[0, 12]$

x	0	2	4	6	8	10	12
$f'(x)$	-19	-21	-25	-28	-29	-28	-25

Estimate $f''(2)$, and estimate $f(8)$.

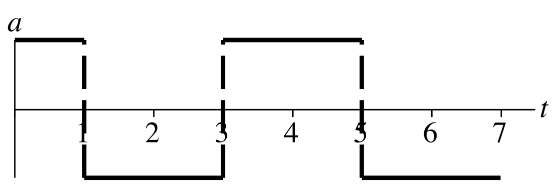
Answer: $f''(2) \approx \dfrac{f'(4) - f'(0)}{4 - 0} = \dfrac{-25 + 19}{4} = -1.5$. By the Fundamental Theorem of Calculus, $f(8) = f(2) + \int_2^8 f'(x)dx = 4 + \int_2^8 f'(x)dx$. Estimating the integral using the trapezoid rule gives -156, so $f(8) \approx 4 - 156 = -152$.

38. The graph of $f'(x)$ is given. Sketch a possible graph for $f(x)$. Mark the points $x_1, \ldots, x_4$ on your graph and label local maxima, local minima, and inflection points on your graph. Note, the left most x_3 in the figure should be x_1.

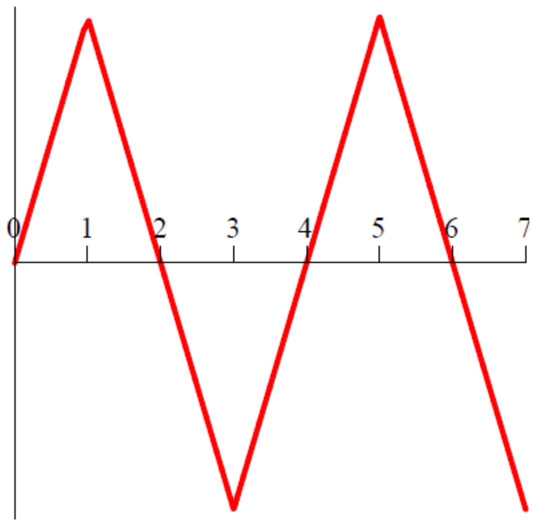


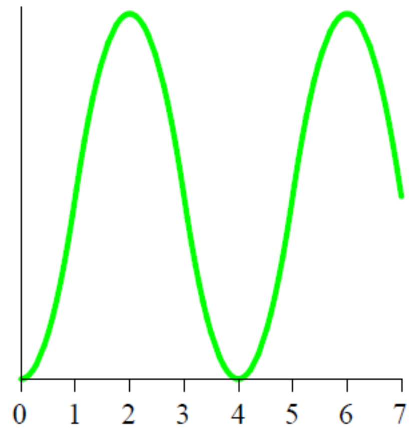
Answer: Note the typo, the first point should have been labelled x_1. $f(x)$ has a local minimum at x_1 (the derivative switches from negative to positive there). $f(x)$ has an inflection point at x_2 (the second derivative switches from positive to negative there), $f(x)$ has a local maximum at x_3 (the derivative switches from positive to negative there). At x_4, the derivative has an inflection point, but that doesn't tell us anything particular about $f(x)$. Your graph should approach $+\infty$ on the left and some kind of horizontal asymptote on the right.

39. The acceleration, a, of a particle as a function of time is shown in the figure. Sketch graphs of velocity and position against time. The particle starts at rest at the origin.



Answer: The velocity must be piecewise linear since accleration is piecewise constant. Also, physically velocity is continuous even if acceleration is not. Since it starts at rest, $v(0) = 0$. So $v(1) = 1$, $v(3) = -1$, etc. Then we integrate again to get position, which is piecewise parabolic (and differentiable everywhere, even though velocity is not). Velocity is graphed in red, and position is in green.





40. An old rowboat has sprung a leak. Water is flowing from the boat at a rate, $r(t)$, given in the following table.

t minutes	0	5	10	15
$r(t)$ liters/min	12	20	24	16

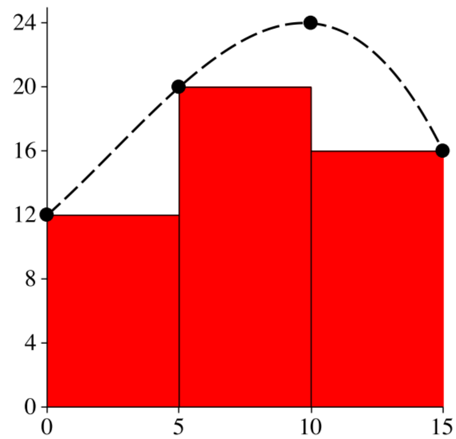
(a) Compute upper and lower estimates for the volume of water that has flowed into the boat during the 15 minutes.
Answer: To get the upper estimate, we use the maximum value over each interval, which is

$$\text{Max Volume} = 5 \cdot (20 + 24 + 24) = 340 \text{ liters.}$$

To get the lower estimate, we use the minimum value over each interval, which is

$$\text{Min Volume} = 5 \cdot (12 + 20 + 16) = 240 \text{ liters.}$$

(b) Draw a graph to illustrate the lower estimate.
Answer: The lower sum is represented by the area of the red rectangles. The data points are represented as dots, and we've connected the dots with a possible curve.



41. Find the antiderivative of $f(x) = 7x^6 - 4x^2 + x$.

    Answer: $x^7 - \frac{4}{3}x^3 + \frac{1}{2}x^2 + C$.

42. Find the antiderivative of $f(x) = 2e^x - 3\sec^2 x$.

    Answer: $2e^x - 3\tan x + C$.

43. Find the antiderivative of $f(x) = 2^x + x^{-\frac{1}{2}} + \sin x$.

    Answer: $\frac{2^x}{\ln 2} - 2x^{\frac{1}{2}} - \cos x + C$.

44. Find the antiderivative of $f(x) = \frac{18x^6+6x^5+3x^3}{x^6}$.

    Answer: $\frac{3x+1}{x^3} + C$.

45. Find the antiderivative F of $f(x) = \sec x \tan x$ such that $F(0) = 101$.

    Answer: $F(x) = \sec x + 100$.

46. Find the antiderivative F of $f(x) = 11e^x - \frac{1}{x}$ such that $F(1) = e$.

    Answer: $F(x) = 11e^x - \ln x - 10e$.

47. Find f if $f''(x) = \cos x + x^5$.

    Answer: $f'(x) = \sin x + \frac{1}{6}x^6 + C$ so $f(x) = -\cos x + \frac{1}{42}x^7 + Cx + C'$