1. The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

$$n = f(t) = \frac{a}{1 + be^{-0.7t}}$$

where t is measured in hours. At time $t = 0$ the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of a and b. According to this model, what happens to the yeast population in the long run?

Solution: $a = 140$ and $b = 6$. In the long run, the population is approaching to 140 cells.

2. Do problem 68 parts a)-e) on page 249-250 of the text.

Solution: $v(t) = 3t^2 - 12$. $a(t) = 6t$. Moving downward for t in $(0, 2)$, moving upward for $t > 2$. To get distance travelled, first find the distance travelled when it is moving downward ($|s(2) - s(0)| = 16$) and add it to the distance travelled when it is moving upward ($s(3) - s(2) = 7$) for a total distance travelled of 23. The particle is speeding up when both velocity and acceleration are positive or both velocity and acceleration are negative. For positive values of t, this occurs when $t > 2$.

3. Do problem 73 on page 250 of the text.

Solution: Find solutions to odd problems in the back of the textbook.

4. A boat at anchor is bobbing up and down in the sea. The vertical distance, y, in feet, between the sea floor and the boat is given as a function of time, t, in minutes by

$$y = 15 + 6\sin(2\pi t)$$

(a) Find $\dfrac{dy}{dt}$.

Solution: $\dfrac{dy}{dt} = 12\pi\cos(2\pi t)$

(b) Find $\dfrac{dy}{dt}$ when $t = \frac{5}{6}$. Explain in an English sentence what this means in terms of the movement of the boat. Include units.

Solution: The derivative at $t = \frac{5}{6}$ is $12\pi\cos\left(\frac{10\pi}{6}\right) = 6\pi$ ft/min. This means that at $t = \frac{5}{6}$ minute ($t = 50$ seconds) the boat is moving upwards at 6π feet/minute.

5. If the position of a particle at time t is given by the formula $s(t) = t^3 - t$, what is the velocity of the particle at time $t = 1$?

Solution: $v = 2$.

6. A rock falling from the top of a vertical cliff drops a distance of $s(t) = 16t^2$ feet in t seconds. What is its speed at time t? What is its speed when it has fallen 64 feet?

Solution: Speed is $32t$ in general, and 64 feet per second when it has fallen 64 feet.

7. The height of a rock at time t which is thrown vertically from a height of 44 feet is given by the formula $s(t) = -t^2 + 20t + 44$. What is the maximum height of the rock? When does it hit the ground? What is the impact speed?

Solution: The maximum height is 144 feet. The rock hits the ground when $t = 22$. The impact speed is 24 feet/sec

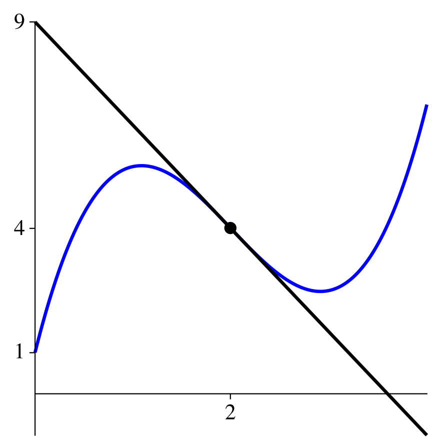
8. By increasing its advertising cost x(in thousands of dollars) for a product, a company discovers that it can increase the sales y(in thousands of dollars) according to the model

$$y = -\frac{1}{10}x^3 + 6x^2 + 400, \ \ 0 \leq x \leq 40.$$

Find the inflection point of this model and interpret in the context of the problem using complete English sentences.

Solution: The model has a point of inflection at $x = 20$. On the interval $(0, 20)$, each additional input returns more than the previous input dollar. By contrast, on the interval $(20, 40)$, each additional dollar of input returns less than the previous input dollar. So an increased investment beyond this point is considered a poor use of capital. This point is called the point of diminishing returns.

9. The Figure shows the tangent line approximation to $f(x)$ near $x = a$.



(a) Find a, $f(a)$, $f'(a)$.

Solution: $a = 2$, $f(a) = 4$, and $f'(a) = -\frac{5}{2}$.

(b) Find an equation for $L(x)$, the tangent line approximation.

Solution: The tangent line is $y = 4 - 2.5(x - 2)$

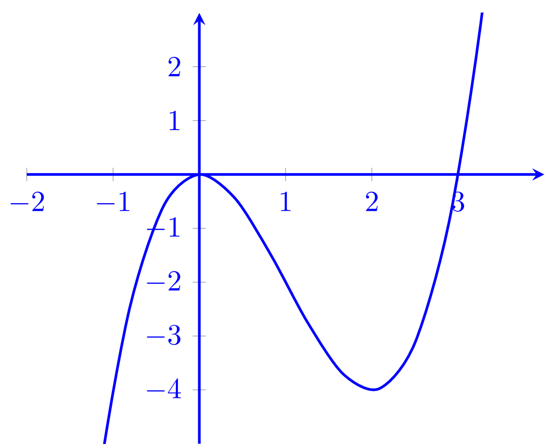
(c) Estimate $f(2.1)$ and $f(1.98)$ using linear approxiation (tangent line approximation). Are these under or overestimates? Which estimate would you expect to be more accurate and why?
Solution: The tangent line is $y = 4 - 2.5(x - 2)$ so that $f(2.1) \approx 3.75$ and $f(1.98) \approx 4.05$. $f(2.1)$ is too small and $f(1.98)$ is too large (by considering concavity of $f(x)$). $L(1.98)$ is more accurate than $L(2.1)$ since 1.98 is closer to 2.

10. Use linear approximation to estimate $\sqrt{24}$.

Solution: $L(x) = 5 + \frac{1}{10}(x - 25)$, so $\sqrt{24} \approx L(24) = 5 + \frac{1}{10}(24 - 25) = 4\frac{9}{10}$.

11. For $f(x) = x^3 - 3x^2$ on $-1 \le x \le 1$, find the critical points of f, the inflection points, the values of f at all these points and the endpoints, and the absolute maxima and minima of f. Then sketch the graph, indicating clearly where f is increasing or decreasing and its concavity.

Solution: $f'(x) = 3x^2 - 6x$. Set equal to 0 and solve to find critical numbers at $x = 0$ and $x = 2$. Using first derivative test, we find the derivative switches from + to - at $x = 0$, so $f(x)$ has a local maximum of $y = 0$ there. The derivative switches from - to + at $x = 2$ so $f(x)$ has a local minimum of $y = -4$ there. $f''(x) = 6x - 6$, the second derivative is zero at $x = 1$. We find that the second derivative switches from - to + at $x = 1$, so $f(x)$ has an inflection point at $(1, -2)$.



Now for the absolute extrema on $[-1, 1]$. Note that the critical number $x = 2$ is not in the interval, so we just need to substitute the endpoints and $x = 0$. $f(-1) = -4$, $f(0) = 0$ and $f(1) = -2$. The absolute maximum on the interval is 0, occurring at $x = 0$ and the absolute minimum is -4, occurring at $x = -1$.

12. For $f(x) = x + \sin x$ on $0 \le x \le 2\pi$, find the critical points of f, the inflection points, the values of f at all these points and the endpoints, and the absolute maxima and minima of f. Then sketch the graph, indicating clearly where f is increasing or decreasing and its concavity.

Solution: The only critical point is $x = \pi$, which is also the only inflection point. Since $f(0) = 0$, $f(\pi) = \pi$, and $f(2\pi) = 2\pi$, the absolute maximum is 2π while the absolute minimum is 0. The function is always increasing and is concave down on $[0, \pi]$ and concave up on $[\pi, 2\pi]$.

13. For $f(x) = \dfrac{4x^2}{x^2+1}$, find the critical points of f, the inflection points, the values of f at all these points, the limits as $x \to \pm\infty$, and the absolute maxima and minima of f. Then sketch the graph, indicating clearly where f is increasing or decreasing and its concavity.

    Solution: The only critical point is $x = 0$, which is a local minimum. The inflection points are at $x = \pm 1/\sqrt{3}$. We have $f(0) = 0$ and $f(\pm 1/\sqrt{3}) = 1$. As $x \to \pm\infty$, $f(x) \to 4$.

14. Find the exact absolute maximum and minimum values of the function

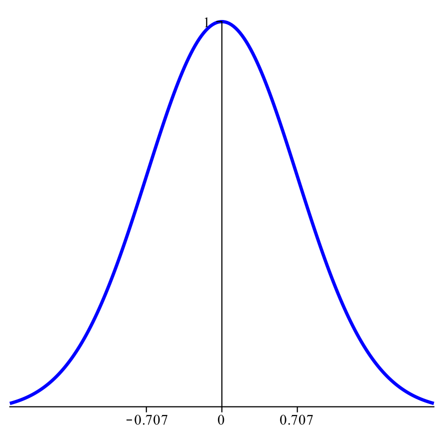
$$h(z) = \frac{1}{z} + 4z^2 \text{ for } z > 0$$

    Solution: There is no absolute maximum. The absolute minimum occurs at $x = \frac{1}{2}$ and the absolute minimum value is 3.

15. Use derivatives to identify local maxima and minima and points of inflection. Then graph the function.

$$f(x) = e^{-x^2}.$$

    Solution: The only local extremum is $x = 0$ which is a local maximum. The inflection points occur at $x = \pm\frac{1}{\sqrt{2}}$. Asymptotically as $x \to \pm\infty$ the function approaches zero.



16. (a) Find all critical points and all inflection points of the function $f(x) = x^4 - 2ax^2 + b$. Assume a and b are positive constants.

Solution: Critical points occur at $x = 0$ and $x = \pm\sqrt{a}$. Inflection points occur at $x = \pm\sqrt{a/3}$.

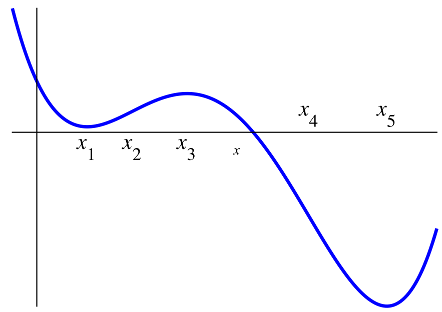
(b) Find values of the parameters a and b if f has a critical point at the point $(2, 5)$.

Solution: If there is a critical point at $(2, 5)$, then $a = 4$ and $b = 21$.

(c) If there is a critical point at $(2, 5)$, where are the inflection points?

Solution: The inflection points happen at $x = \pm\sqrt{4/3}$.

17. For the function, f, graphed in the Figure:

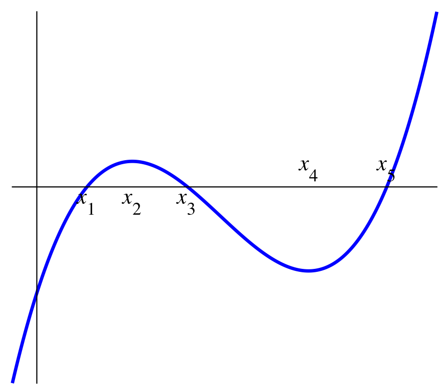


(a) Sketch $f'(x)$.

(b) Where does $f'(x)$ change its sign?

(c) Where does $f'(x)$ have local maxima or minima?

Solution: Here is the derivative:



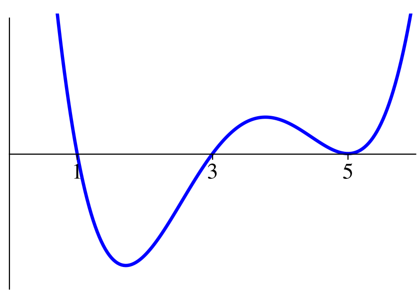
$f'(x)$ changes sign at x_1, x_3, and x_5. It has a local maximum at x_2 and a local minimum at x_4.

18. Using your answer to the previous problem as a guide, write a short paragraph (using complete sentences) which describes the relationships between the following features of a function f:

- The local maxima and minima of f.
- The points at which the graph of f changes concavity.
- The sign changes of f'.
- The local maxima and minima of f'.

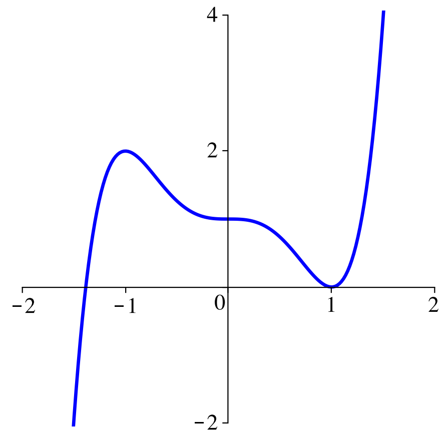
Solution: You answer should include the points that the local maxima and minima of f are exactly the sign changes of f'. A local minimum occurs where $f'(x)$ switches from - to +, while a local maximum occurs where $f'(x)$ switches from + to -. The points at which the graph of f changes concavity are exactly the local maxima and local minima of f', because this is where $f''(x)$ switches sign.

19. The figure shows a graph of $y = f'(x)$ (not the function f). For what values of x does f have a local maximum? A local minimum?



Solution: f has a local maximum at $x = 1$ and a local minimum at $x = 3$. The point at $x = 5$ is a stationary point which is neither a local maximum nor a local minimum.

20. On the graph of f' shown in the figure, indicate the x-values that are critical points of the function f itself. Are they local minima, local maxima, or neither?

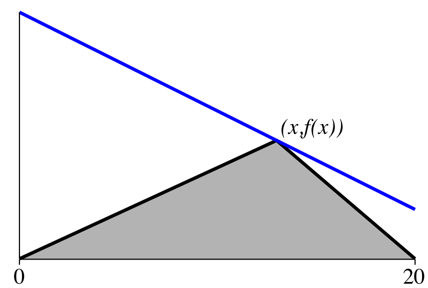


Solution: The critical points of f occur at $x = -1.5$ and $x = 1$. The point at $x = -1.5$ is a local minimum, and the point at $x = 1$ is neither.

21. Find a formula for the function described: a cubic polynomial with a local maximum at $x = 1$, a local minimum at $x = 3$, a y-intercept of 5, and an x^3 term whose coefficient is 1.

Solution: It must be of the form $y = x^3 + ax^2 + bx + c$ for some constants a, b, and c. The information given tells us $c = 5$, $3 + 2a + b = 0$, and $27 + 6a + b = 0$, so that $a = -6$ and $b = 9$.

22. Find the x-value maximizing the shaded area. One vertex is on the graph of $f(x) = x^2/3 - 50x + 1000$ (shown in blue).



Solution: The area of the region is $A = 10(x^2/3 - 50x + 1000)$. The critical point is at $x = 75$, which is not in the domain $0 \leq x \leq 20$. At the endpoint $x = 0$ we have $A = 10,000$, and at the endpoint $x = 20$ we have $A = \frac{4,000}{3}$. So the maximum is at $x = 0$.

23. Given that the surface area of a closed cylinder of radius r cm and height h cm is 8 cm^2, find the dimensions giving the maximum volume.

Solution: The surface area is $S = 2\pi rh + 2\pi r^2 = 8$, so that $h = \frac{4}{\pi r} - r$. The volume is

$$V = \pi r^2 h = 4r - \pi r^3$$

which has a critical point at $r = \frac{2}{\sqrt{3\pi}}$. The endpoints are $r = 0$ (which gives zero volume) and $h = 0$ (which also gives no volume), so this critical point must be the maximizer. At this critical point we have $h = \frac{4}{\sqrt{3\pi}}$.

24. Do the last two problems from Project 11.

Solution: Solutions are posted on the "Projects" page of the website.

25. Find the range of the function $f(x) = x^3 - 6x^2 + 9x + 5$, if the domain is $0 \leq x \leq 5$.

Solution: The range is $[5, 25]$, determined by finding the absolute extrema on the domain, and the fact that $f(x)$ is a continuous function.

26. A landscape architect plans to enclose a 3000 square foot rectangular region in a botanical garden. She will use shrubs costing \$45 per foot along three sides and fencing costing \$20 per foot along the fourth side. Find the minimum total cost.

    Solution: The area constraint is $xy = 3000$, and the fencing cost is $C = 90x + 65y$. The optimum values are $x = 46.5$ and $y = 64.5$ feet.

27. A piece of wire of length L cm is cut into two pieces. One piece, of length x cm, is made into a circle; the rest is made into a square. Find the value of x that makes the sum of the areas of the circle and square a minimum. Find the value of x giving a maximum.

    Solution: Let x be the amount of wire used for the circle. Then the perimeter of the square is $L - x$, so the side length of the square is $y = \frac{L-x}{4}$. The radius of the circle is $r = \frac{x}{2\pi}$. The sum of the areas is

    $$A = \pi r^2 + y^2 = \frac{x^2}{4\pi} + \frac{(L-x)^2}{16}.$$

    The critical point is at $x = \frac{L\pi}{\pi+4} = 0.44L$.

28. Water is being pumped into a vertical cylinder of radius 5 meters and height 20 meters at a rate of 3 meters^3/min. How fast is the water level rising when the cylinder is half full?

    Solution: It is rising at 0.0382 meters per minute.

29. Does L'Hopital's rule apply to the limit? If so, evaluate it.

    $$\lim_{x\to 0} \frac{x}{e^x}.$$

    Solution: No. The numerator is going to 0 but the denominator is going to 1. So the limit is 0 without using L'Hopital's rule.

30. Does L'Hopital's rule apply to the limit? If so, evaluate it.

    $$\lim_{x\to 1} \frac{\sin \pi x}{x-1}.$$

    Solution: Yes. Both the numerator and denominator are approaching zero.

    $$\lim_{x\to 1} \frac{\sin \pi x}{x-1} = \lim_{x\to 1} \frac{\pi \cos \pi x}{1} = -\pi.$$

31. Determine whether the limit exists, and if possible evaluate it.

$$\lim_{t\to 0} \frac{e^t - 1 - t}{t^2}$$

Solution: The limit approaches $\frac{1}{2}$, by applying L'Hopital's rule twice.

32. Determine whether the limit exists, and if possible evaluate it.

$$\lim_{t\to 0^+} \frac{3\sin t - \sin 3t}{3\tan t - \tan 3t}$$

Solution: It is of the form $\frac{0}{0}$. So we use L'Hopital's rule to get

$$\begin{aligned}
\lim_{t\to 0^+} \frac{3\sin t - \sin 3t}{3\tan t - \tan 3t} &= \lim_{t\to 0^+} \frac{3\cos t - 3\cos 3t}{3\sec^2 t - 3\sec^2 3t} = \frac{0}{0} \\
&= \lim_{t\to 0^+} \frac{-3\sin t + 9\sin 3t}{6\sec^2 t\tan t - 18\sec^2 3t\tan 3t} = \frac{0}{0} \\
&= \lim_{t\to 0^+} \frac{-3\cos t + 27\cos 3t}{6\sec^4 t + 12\sec^2 t\tan^2 t - 54\sec^4 3t - 108\sec^2 3t\tan^2 3t} \\
&= \frac{24}{-48} = -\frac{1}{2}.
\end{aligned}$$

33. True or false? If true, explain how you know. If false, give a counterexample. If $f'(p) = 0$, then $f(x)$ has a local minimum or local maximum at $x = p$.

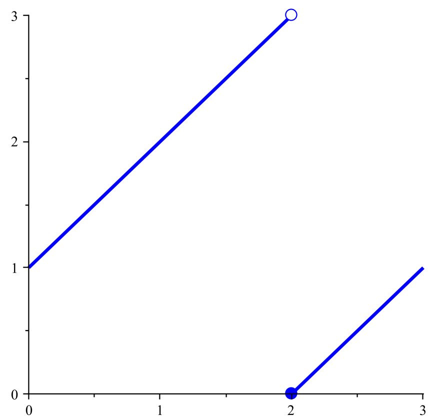
Solution: False. Let $f(x) = x^3$ and $p = 0$. Then $f'(0) = 0$, but $x = 0$ is neither a local minimum nor a local maximum.

34. True or false? If true, explain how you know. If false, give a counterexample. If $f''(p) = 0$, then the graph of f has an inflection point at $x = p$.

Solution: False. Let $f(x) = x^4$ and $p = 0$. Then $f''(0) = 0$, but $f''(x) = 12x^2$ which never changes sign.

35. Sketch the graph of a function defined at every point of $[0, 2]$ which has an absolute minimum but no absolute maximum. Why does your graph not contradict the Extreme Value Theorem?

    Solution: Here is one.

    The global minimum happens at $x = 2$, but there is no global maximum. The Extreme Value Theorem only applies to *continuous* functions, and this one is not.

36. Do problem 68 parts a)-e) on page 249-250 of the text.

    Solution: $v(t) = 3t^2 - 12$. $a(t) = 6t$. Moving downward for t in $(0, 2)$, moving upward for $t > 2$. To get distance travelled, first find the distance travelled when it is moving downward ($|s(2) - s(0)| = 16$) and add it to the distance travelled when it is moving upward ($s(3) - s(2) = 7$) for a total distance travelled of 23. The particle is speeding up when both velocity and acceleration are positive or both velocity and acceleration are negative. For positive values of t, this occurs when $t > 2$.

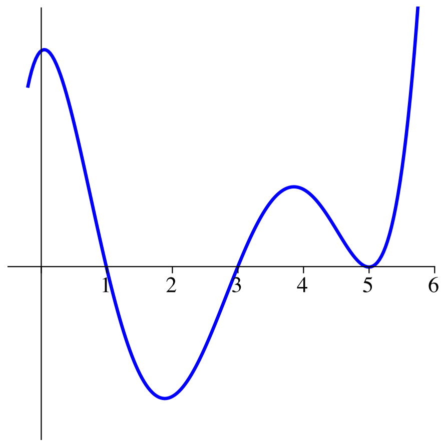
37. Do problem 73 on page 250 of the text.

    Solution: Find solutions to odd problems in the back of the textbook.

38. Precisely state the Mean Value Theorem.

    Solution: See page 272 of the text.

39. The graph of $y = f''(x)$ is shown. Where are the inflection points of $y = f(x)$?



Solution: The inflection points are at $x = 1$ and $x = 3$, where the sign of $f''(x)$ changes. $x = 5$ is not an inflection point.

40. Find the absolute maximum and minimum of the function $f(x) = 9x^{1/3} - x^3$ on the closed interval $[-1, 8]$.

Solution: The absolute maximum is 8, occurring at $x = 1$, and the absolute minimum is -494, occuring at $x = 8$.

41. Which point on the parabola $y = x^2 - x$ minimizes the square of the distance to the point $(1, 1)$?
Solution: The square of the distance is

$$D = (x - 1)^2 + (x^2 - x - 1)^2.$$

Critical points happen at $x = 0$ and $x = \frac{3}{2}$. We compute $D(0) = 2$ and $D(\frac{3}{2}) = \frac{5}{16}$, so the minimizer is $(\frac{3}{2}, \frac{3}{4})$.

42. Compute
$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n.$$

Solution: e^2.

43. Compute
$$\lim_{x\to 1}\frac{\ln x}{x^2-1}.$$

Solution: $\frac{1}{2}$.

44. Compute
$$\lim_{x\to 2}\frac{x^2-3x+2}{x+5}.$$

Solution: 0 .

45. Let f be a continuous function with domain $[-1,3]$, and assume that f is differentiable on $(-1,3)$. Which of the following statements about f are definitely true?

(a) If f has a critical point at $x=2$, then f has either a local maximum or a local minimum at $x=2$.

(b) If f has a critical point at $x=1$ and $f''(x)<0$, then f has a local maximum at $x=1$.

(c) If f has a critical point at $x=1$ and $f''(x)<0$, then f has an absolute maximum at $x=1$.

(d) If $f(3)>f(-1)$, then f has an absolute maximum at $x=3$.

(e) If $f'(0)=0$ and $f(0)>f(3)>f(-1)$, then f has an absolute maximum at $x=0$.

(f) If the only critical point of f occurs at $x=2$, then f must have an absolute maximum at either $x=-1,2$, or 3.

(g) It is possible that f might not have an absolute minimum on this domain.

(h) If f has a local minimum at $x=2$ then $f'(2)=0$.

(i) If $f''(x)>0$ for all values of x in the domain, then $f(3)\geq f(-1)$.

(j) If $f''(x)>0$ for all values of x in the domain, then $f'(3)\geq f'(-1)$.

(k) If f has an inflection point at $x=1$ then $f'(x)$ has either a local maximum or a local minimum at $x=1$.

(l) If $f''(1)=0$ then f has an inflection point at $x=1$.

Solution: True: (b), (f), (h), (j), (k) False: (a), (c), (d), (e), (g), (i), (l)

Derivative Practice.

Find $f'(x)$.

1. $f(x) = \dfrac{e^x}{\ln(x)}$

   Solution: $f'(x) = \dfrac{(\ln(x) - 1/x)e^x}{(\ln(x))^2}$

2. $f(x) = \ln(\cos(x))$

   Solution: $f'(x) = -\tan(x)$

3. $f(x) = \arctan(\ln(\cos(x^2)))$

   Solution: $\dfrac{-2x\sin(x^2)}{(1 + (\ln(\cos(x^2)))^2)\cos(x^2)}$

4. $f(x) = \arcsin(\sin(x))$

   Solution: $f'(x) = 1$

5. $f(x) = \arctan(4x + 3)$

   Solution: $f'(x) = \dfrac{4}{1 + (4x + 3)^2}$

6. $f(x) = (\arcsin(x))(\arctan(x))$

   Solution: $f'(x) = \dfrac{\arctan x()}{\sqrt{1 - x^2}} + \dfrac{\arcsin(x)}{1 + x^2}$

7. $f(x) = \arcsin(x \tan(x))$

Solution: $f'(x) = \dfrac{\tan x + x \sec^2 x}{\sqrt{1 - x^2 \tan^2(x)}}$

8. $f(x) = e^{\arcsin(4x^2)}$

Solution: $f'(x) = \dfrac{8xe^{\arcsin(4x^2)}}{\sqrt{1 - 16x^4}}$

9. $f(x) = \ln(\arcsin(x)) + xe^{x^2}$

Solution: $f'(x) = \dfrac{1}{\arcsin(x)\sqrt{1 - x^2}} + (1 + 2x^2)e^{x^2}$

10. $f(x) = \tan^2(\arcsin(1))$

Solution: $f'(x) = 0$

11. $f(x) = \sec(\ln(x))$

Solution: $f'(x) = \dfrac{\sec(\ln(x))\tan(\ln(x))}{x}$

12. $f(x) = x^{\ln(x)}$

Solution: $f'(x) = x^{\ln(x)} \dfrac{2\ln(x)}{x}$