Review Sheet for Second Midterm
Mathematics 1300, Calculus 1

1. For what values of x is the graph of $y = x^5 - 5x$ both increasing and concave up?

   Answer: $x > 1$.

2. Where does the tangent line to $y = 2^x$ through $(0, 1)$ intersect the x-axis?

   Answer: $\left(-\frac{1}{\ln 2}, 0\right)$.

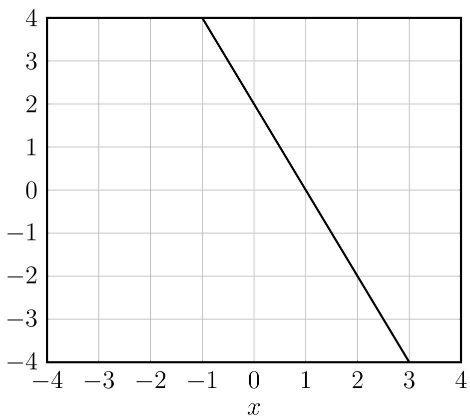
3. If $g(x) = e^x f(x)$, find and simplify $g''(x)$.

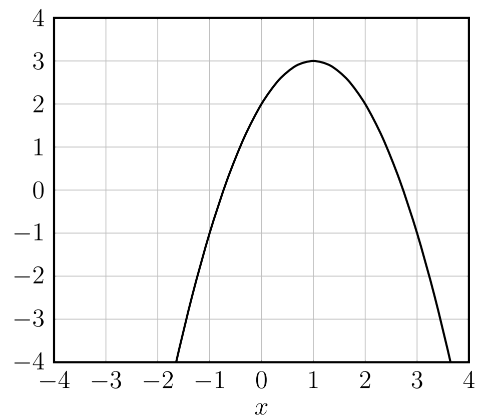
   Answer: $g''(x) = e^x\big(f(x) + 2f'(x) + f''(x)\big)$.

4. If $f(x) = 4x^3 + 6x^2 - 23x + 7$, find the intervals on which $f'(x) \geq 1$.

   Answer: $[1, \infty)$ and $(-\infty, -2]$.

5. Let $f(x)$ and $g(x)$ be the functions graphed below.





   Let $h(x) = f(x)g(x)$, let $j(x) = x^2 f(x)$, let $k(x) = f(x^2)$, let $p(x) = f(x)/g(x)$, let $q(x) = f(g(x))$, and $r(x) = g(g(x))$.

   (a) Estimate $h'(1)$, $h'(0)$, $p'(0)$, $q'(0)$, $r'(1)$, and $r'(2)$.

   Answer: $h'(1) = -6$, $h'(0) = 0$, $p'(0) = -2$, $q'(0) = -4$, $r'(1) = 0$, $r'(2) = 4$.

   (b) Estimate $j'(-1)$ and $k'(-1)$.

   Answer: $j'(-1) = -10$ and $k'(-1) = 4$.

   (c) Estimate all values of x for which $y = r(x)$ has a horizontal tangent line.

   Answer: $x = 1$, $x = -0.4$, $x = 2.4$.

6. Using the information in the table, find:

(a) $h(4)$ if $h(x) = f(g(x))$

(b) $h'(4)$ if $h(x) = f(g(x))$

(c) $h(4)$ if $h(x) = g(f(x))$

(d) $h'(4)$ if $h(x) = g(f(x))$

(e) $h'(4)$ if $h(x) = g(x)/f(x)$

(f) $h'(4)$ if $h(x) = f(x)g(x)$

x	1	2	3	4
$f(x)$	3	2	1	4
$f'(x)$	1	4	2	3
$g(x)$	2	1	4	3
$g'(x)$	4	2	3	1

Answer:

(a) $h(4) = 1$

(b) $h'(4) = 2$

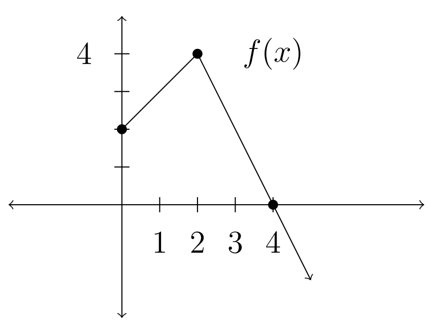
(c) $h(4) = 3$

(d) $h'(4) = 3$

(e) $h'(4) = -\frac{5}{16}$

(f) $h'(4) = 13$

7. A graph of $f(x)$ is shown below. It is piecewise linear. The table below gives values of $g(x)$ and $g'(x)$.



x	0	1	2	3	4
$g(x)$	2	5	9	11	8
$g'(x)$	3	4	2	-3	-4

a) Given $h(x) = f(x)g(x)$, find $h'(1)$.

b) Given $k(x) = \dfrac{f(x)}{g(x)}$, find $k'(3)$.

c) Given $\ell(x) = \dfrac{g(x)}{\sqrt{x}}$, find $\ell'(4)$.

d) Given $m(x) = g(f(x))$, find $m'(3)$.

Answer: $h'(1) = 17$, $k'(3) = -\dfrac{16}{121}$, $\ell'(4) = -\dfrac{5}{2}$, $m'(3) = -4$.

8. On what interval(s) is the function

$$f(x) = \frac{(5x+2)^3}{(2x+3)^5}$$

increasing?

Answer: $(-\infty, -\frac{3}{2}) \cup (-\frac{3}{2}, \frac{5}{4})$.

9. If $g(2) = 3$ and $g'(2) = -4$, find $f'(2)$ for the following:

   (a) $f(x) = x^2 - 4g(x)$
   (b) $f(x) = \frac{x}{g(x)}$
   (c) $f(x) = x^2 g(x)$
   (d) $f(x) = g(x)^2$
   (e) $f(x) = x \sin(g(x))$
   (f) $f(x) = x^2 \ln(g(x))$

Answer:

   (a) $f'(2) = 20$
   (b) $f'(2) = \frac{11}{9}$
   (c) $f'(2) = -4$
   (d) $f'(2) = -24$
   (e) $f'(2) = \sin 3 - 8\cos 3$
   (f) $f'(2) = 4\ln 3 - \frac{16}{3}$

10. On what interval(s) is the function

$$f(x) = (x+3)e^{2x}$$

decreasing? On what intervals is it concave down?

Answer: Decreasing on $(-\infty, -7/2)$. Concave down on $(-\infty, -4)$.

11. If the position of a particle at time t is given by the formula $s(t) = t^3 - t$, what is the velocity of the particle at time $t = 1$?

Answer: $v = 2$.

12. A rock falling from the top of a vertical cliff drops a distance of $s(t) = 16t^2$ feet in t seconds. What is its speed at time t? What is its speed when it has fallen 64 feet?

Answer: Speed is $32t$ in general, and 64 feet per second when it has fallen 64 feet.

13. The height of a rock at time t which is thrown vertically on the height 46 feet is given by the formula $s(t) = -t^2 + 20t + 44$. What is the maximum height of the rock? When does it hit the ground? What is the impact speed?

Answer: The maximum height is 144 feet. The rock hits the ground when $t = 22$. The impact speed is 4 feet/sec

14. By increasing its advertising cost x(in thousands of dollars) for a product, a company discovers that it can increase the sales y(in thousands of dollars) according to the model

$$y = -\frac{1}{10}x^3 + 6x^2 + 400, \quad 0 \leq x \leq 40.$$

Find the inflection point of this model and interpret in the context of the problem using complete English sentences.

Answer: The model has a point of inflection at $x = 20$. On the interval $(0, 20)$, each additional input returns more than the previous input dollar. By contrast, on the interval $(20, 40)$, each additional dollar of input returns less than the previous input dollar. So an increased investment beyond this point is considered a poor use of capital. This point is called the point of diminishing returns.

15. If $f(x) = x^2 + 1$ and $g(x) = 5 - x$, find:

(a) $h'(1)$ if $h(x) = f(x) \cdot g(x)$

(b) $j'(2)$ if $j(x) = \frac{f(x)}{g(x)}$

(c) $k'(3)$ if $k(x) = f(g(x))$

Answer:

(a) $h'(1) = 6$

(b) $j'(2) = \frac{17}{9}$

(c) $k'(3) = -4$

16. A boat at anchor is bobbing up and down in the sea. The vertical distance, y, in feet, between the sea floor and the boat is given as a function of time, t, in minutes by

$$y = 15 + 6\sin(2\pi t)$$

(a) Find $\frac{dy}{dt}$. **Answer:** $\frac{dy}{dt} = 12\pi\cos(2\pi t)$

(b) Find $\frac{dy}{dt}$ when $t = \frac{5}{6}$. Explain in an English sentence what this means in terms of the movement of the boat. Include units.

Answer: The derivative at $t = \frac{5}{6}$ is $12\pi\cos\left(\frac{10\pi}{6}\right) = 6\pi$ ft/min. This means that at $t = \frac{5}{6}$ minute ($t = 50$ seconds) the boat is moving upwards at 6π feet/minute.

17. Find $\frac{dy}{dx}$ if

$$x^3 + y^3 - 4x^2y = 0.$$

Answer: $\dfrac{dy}{dx} = \dfrac{8xy - 3x^2}{3y^2 - 4x^2}$.

18. Consider the function $y + \sin y + x^2 = 9$.

(a) Find the slope of the curve at $(3, 0)$.

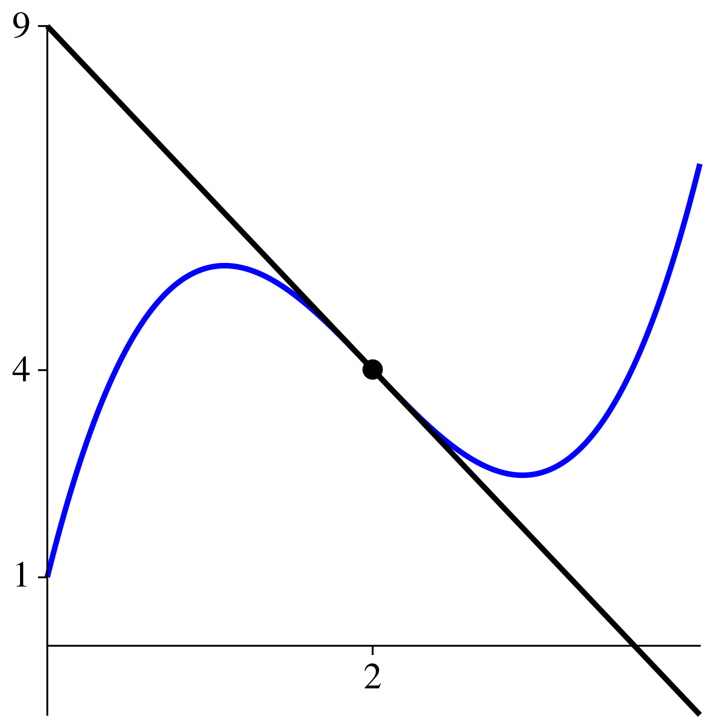
(b) Use linear approximation to estimate the y-coordinate when $x = 2.8$.

Answer:

(a) The slope is -3.

(b) The approximate y-value is 0.6.

19. The Figure shows the tangent line approximation to $f(x)$ near $x = a$.



(a) Find a, $f(a)$, $f'(a)$.

(b) Find an equation for $L(x)$, the tangent line approximation.

(c) Estimate $f(2.1)$ and $f(1.98)$ using linear approxiation (tangent line approximation). Are these under or overestimates? Which estimate would you expect to be more accurate and why?

Answer: $a = 2$, $f(a) = 4$, and $f'(a) = -\frac{5}{2}$. The tangent line is $y = 4 - 2.5(x-2)$ so that $f(2.1) \approx 3.75$ and $f(1.98) \approx 4.05$. $f(2.1)$ is too small and $f(1.98)$ is too large (by considering concavity of $f(x)$). $f(1.98)$ is more accurate that $f(2.1)$ since 1.98 is closer to 2.

20. Use linear approximation to estimate $\sqrt{24}$. **Answer:** $L(x) = 5 + \frac{1}{10}(x - 25)$, so $\sqrt{24} \approx L(24) = 5 + \frac{1}{10}(24 - 25) = 4\frac{9}{10}$.

21. Find the equations for the lines tangent to the graph of $xy + y^2 = 4$ when $x = 3$.

Answer: The equations are

$$y - 1 = -\frac{1}{5}(x-3) \qquad \text{and} \qquad y + 4 = -\frac{4}{5}(x-3).$$

22. Consider the curve $x^2 + 2xy + 5y^2 = 4$. At what point(s) is the tangent line to this curve horizontal? At what point(s) is the tangent line to this curve vertical? At what points is the slope of the tangent line equal to 2?

Answer: The derivative is $\dfrac{dy}{dx} = -\dfrac{x+y}{x+5y}$. The tangent line is horizontal when $y = -x$, which happens at the points $(1, -1)$ and $(-1, 1)$. The tangent line is vertical when $x = -5y$, which happens at the points $(\frac{\sqrt{5}}{5}, -\sqrt{5})$ and $(-\frac{\sqrt{5}}{5}, \sqrt{5})$. The slope of the tangent line equals 2 when $x = -\frac{11}{3}y$, which happens at the points $(-\frac{11}{5}, \frac{3}{5})$ and $(\frac{11}{5}, -\frac{3}{5})$.

23. Ice is being formed in the shape of a circular cylinder with inner radius 1 cm and height 3 cm. The outer radius of the ice is increasing at 0.03 cm per hour when the outer radius is 1.5 cm. How fast is the volume of the ice increasing at this time?

Answer: Assuming the inner radius and height are not changing, the volume is

$$V = 3\pi(r^2 - 1).$$

So when $r = 1.5$ and $dr/dt = 0.03$, we get

$$\frac{dV}{dt} = 0.848 \text{ cm}^3/hr.$$

24. Water is being pumped into a vertical cylinder of radius 5 meters and height 20 meters at a rate of 3 meters^3/min. How fast is the water level rising when the cylinder is half full? **Answer:** It is rising at 0.0382 meters per minute.

25. A chemical storage tank is in the shape of an inverted cone with depth 12 meters and top radius 5 meters. When the depth of the chemical in the tank is one meter, the level is falling at 0.1 meters per minute. How fast is the volume of chemical changing? (The volume of a cone of radius r and height h is $V = \frac{\pi}{3}r^2h$.)

Answer: It is decreasing by 0.0545 cubic meters per minute.

26. A spherical balloon is inflated so that its radius is increasing at a constant rate of 1 cm per second. At what rate is air being blown into the balloon when its radius is 5 cm? (The volume of a sphere of radius r is $V = \frac{4\pi}{3}r^3$.)

    Answer: $\frac{dV}{dt} = 100\pi \frac{\text{cm}^3}{\text{sec}}$

27. A radio navigation system used by aircraft gives a cockpit readout of the distance, s, in miles, between a fixed ground station and the aircraft. The system also gives a readout of the instantaneous rate of change, ds/dt, of this distance in miles/hour. An aircraft on a straight flight path at a constant altitude of 10,560 feet (2 miles) has passed directly over the ground station and is now flying away from it. What is the speed of this aircraft along its constant altitude flight path when the cockpit readouts are $s = 4.6$ miles and $ds/dt = 210$ miles/hour?

    Answer: s is measured along the hypotenuse of the right triangle; we are given ds/dt and we want to find dx/dt, where $x^2 + 2^2 = s^2$. Differentiating with respect to time we get

    $$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$

    so that

    $$\frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt} = \frac{s}{\sqrt{s^2-4}}\frac{ds}{dt}.$$

    When $s = 4.6$ and $ds/dt = 210$, we get

    $$\frac{dx}{dt} = 233 \text{ miles per hour.}$$

28. A gas station stands at the intersection of a north-south road and an east-west road. A police car is traveling toward the gas station from the east, chasing a stolen truck which is traveling north away from the gas station. The speed of the police car is 100 mph at the moment it is 3 miles from the gas station. At the same time, the truck is 4 miles from the gas station going 80 mph. At this moment, is the distance between the car and truck increasing or decreasing? How fast? (Distance is measured along a straight line joining the car and truck.)

    Answer: The distance is increasing at 4 miles per hour.

29. The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

    $$n = f(t) = \frac{a}{1 + be^{-0.7t}}$$

    where t is measured in hours. At time $t = 0$ the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of a and b. According to this model, what happens to the yeast population in the long run?

    Answer: $a = 140$ and $b = 6$. In the long run, the population is approaching to 140 cells.

30. Do problem 68 parts a)-e) on page 249-250 of the text.

    Answer: $v(t) = 3t^2 - 12$. $a(t) = 6t$. Moving downward for t in $(0, 2)$, moving upward for $t > 2$. To get distance travelled, first find the distance travelled when it is moving downward ($|s(2) - s(0)| = 16$) and add it to the distance travelled when it is moving upward ($s(3) - s(2) = 7$) for a total distance travelled of 23. The particle is speeding up when both velocity and acceleration are positive or both velocity and acceleration are negative. For positive values of t, this occurs when $t > 2$.

31. Do problem 73 on page 250 of the text.

    Answer: Find solutions to odd problems in the back of the textbook.

Derivative Practice.

Find $f'(x)$.

1. $f(x) = \dfrac{x^2+1}{5}$

   Answer: $f'(x) = \dfrac{2}{5}x$

2. $f(x) = \pi^3$

   Answer: $f'(x) = 0$

3. $f(x) = \sqrt{x} + \frac{1}{3x}$

   Answer: $f'(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{3x^2}$

4. $f(x) = \dfrac{1+x^2+x^3+x^4+x^5+x^6}{x^3}$

   Answer: $f'(x) = -\dfrac{3}{x^4} - \dfrac{1}{x^2} + 1 + 2x + 3x^2$

5. $f(x) = \dfrac{1}{x^2}$

   Answer: $f'(x) = -\dfrac{2}{x^3}$

6. $f(x) = (x+1)(2x-1)$

   Answer: $f'(x) = 4x+1$

7. $f(x) = xe^x$

   Answer: $f'(x) = (x+1)e^x$

8. $f(x) = \sin x \cos x$

   Answer: $f'(x) = \cos^2 x - \sin^2 x$

9. $f(x) = (\ln x)e^x$

   Answer: $f'(x) = e^x(\ln x + 1/x)$

10. $f(x) = kx^n$

    Answer: $f'(x) = nkx^{n-1}$

11. $f(x) = \dfrac{2x-1}{x+3}$

    Answer: $f'(x) = \dfrac{7}{(x+3)^2}$

12. $f(x) = (2x^7 - x^2) \cdot \dfrac{x-1}{x+1}$

    Answer: $f'(x) = \dfrac{2x(7x^7+2x^6-7x^5-x^2-x+1)}{(x+1)^2}$

13. $f(x) = \dfrac{x^2+1}{x+1}$

    Answer: $f'(x) = \dfrac{x^2+2x-1}{(x+1)^2}$

14. $f(x) = \frac{\sec(x)}{1+\tan x}$

    Answer: $\dfrac{(1+\tan x)(\sec x \tan x) - \sec^3 x}{(1+\tan x)^2}$

15. $f(x) = 2\sin^2(x)$

    Answer: $f'(x) = 4\sin x \cos x$

16. $f(x) = \cos^2(x) + \sin^2(x)$

    Answer: $f'(x) = 0$

17. $f(x) = \dfrac{\sin x \cos x}{1 + x\tan x}$

    Answer: $f'(x) = \dfrac{\cos^2 x - \sin^2 x}{1 + x\tan x} - \dfrac{\sin^2 x + x\tan x}{(1+x\tan x)^2}$

18. $f(x) = x^2 \cos x + 4\sin x$

    Answer: $f'(x) = (2x+4)\cos x - x^2 \sin x$

19. $f(x) = \dfrac{e^x}{\ln x}$

    Answer: $f'(x) = \dfrac{(\ln x - 1/x)e^x}{(\ln x)^2}$

20. $f(x) = (x^2+1)^{1000}$

Answer: $f'(x) = 2000x(x^2+1)^{999}$

21. $f(x) = \ln(\cos x)$

Answer: $f'(x) = -\tan x$

22. $f(x) = (4x^2+1)^5$

Answer: $f'(x) = 40x(4x^2+1)^4$

23. $f(x) = \arctan \ln \cos x^2$

Answer: $\dfrac{-2x\sin x^2}{(1+(\ln\cos x^2)^2)\cos x^2}$

24. $f(x) = e^{x^2+x+1}$

Answer: $f'(x) = (2x+1)e^{x^2+x+1}$

25. $f(x) = \sin(\cos(\tan x))$

Answer: $f'(x) = -\cos(\cos(\tan x))\sin(\tan x)\sec^2 x$

26. $f(x) = \sqrt{x^3-2x+5}$

Answer: $f'(x) = \frac{1}{2}(x^3-2x+5)^{-1/2}(3x^2-2)$

27. $f(x) = \dfrac{1}{(x^5-x+1)^9}$

Answer: $f'(x) = -9(x^5-x+1)^{-10}(5x^4-1)$

28. $f(x) = \dfrac{(2x+3)^3}{(4x^2-1)^8}$

Answer: $f'(x) = -\dfrac{2(2x+3)^2(52x^2+96x+3)}{(4x^2-1)^9}$

29. $f(x) = x^5\cos\left(\frac{1}{x}\right)$

Answer: $f'(x) = 5x^4\cos\left(\frac{1}{x}\right) + x^3\sin\left(\frac{1}{x}\right)$

30. $f(x) = \arcsin(\sin x)$

Answer: $f'(x) = 1$

31. $f(x) = \arctan(4x+3)$

Answer: $f'(x) = \dfrac{4}{1+(4x+3)^2}$

32. $f(x) = (\arcsin x)(\arctan x)$

Answer: $f'(x) = \dfrac{\arctan x}{\sqrt{1-x^2}} + \dfrac{\arcsin x}{1+x^2}$

33. $f(x) = \arcsin(x\tan x)$

Answer: $f'(x) = \dfrac{\tan x + x\sec^2 x}{\sqrt{1-x^2\tan^2 x}}$

34. $f(x) = e^{\arcsin(4x^2)}$

Answer: $f'(x) = \dfrac{8xe^{\arcsin(4x^2)}}{\sqrt{1-16x^4}}$

35. $f(x) = \ln(\arcsin x) + xe^{x^2}$

Answer: $f'(x) = \dfrac{1}{\arcsin x\sqrt{1-x^2}} + (1+2x^2)e^{x^2}$

36. $f(x) = \tan^2(\arcsin 1)$

Answer: $f'(x) = 0$

37. $f(x) = \dfrac{1+x}{1-x}$

Answer: $f'(x) = \dfrac{2}{(x-1)^2}$

38. $f(x) = \sec\ln x$

Answer: $f'(x) = \dfrac{\sec\ln x\tan\ln x}{x}$

39. $f(x) = \sqrt{x+\sqrt{x+\sqrt{x}}}$

Answer: $\dfrac{1+\dfrac{1}{2\sqrt{x+\sqrt{x}}}\cdot\left(1+\frac{1}{2\sqrt{x}}\right)}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}$

40. $f(x) = x^{\ln x}$

Answer: $f'(x) = x^{\ln x}\dfrac{2\ln x}{x}$