4.4 APPLIED OPTIMIZATION.

GOAL: FIND A FUNCTION THAT MODELS A PROBLEM, USE THE TECHNIQUES OF 4.1 & 4.2 TO FIND THE "BEST" VALUE.

SUGGESTED PROCEDURE:

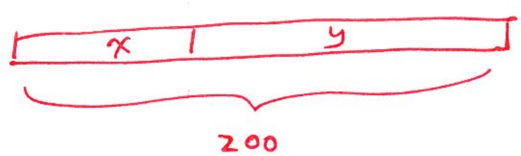
1. DRAW A DIAGRAM, LABEL VARIABLES
2. IDENTIFY QUANTITY TO MAXIMIZE/MINIMIZE
3. FIND A FORMULA FOR THE QUANTITY TO MINIMIZE/MAXIMIZE
4. USE CONSTRAINTS TO ELIMINATE EXTRA VARIABLES. (i.e., GET A FUNCTION OF ONE VARIABLE)
5. FIND THE DOMAIN
6. FIND THE GLOBAL MIN/MAX (DON'T FORGET TO ANSWER THE QUESTION) (FIND THE DERIVATIVE, FIND THE C.P.S, THEN EITHER
   a. IF THE INTERVAL IS CLOSED: SUBSTITUTE ENDPOINTS AND C.PS INTO FUNCTION, CHOOSE LARGEST/SMALLES

   OR

   b. IF THE INTERVAL IS OPEN: HOPE THERE IS ONLY ONE C.P., SHOW THERE IS A LOCAL MAX/MIN THERE, CONCLUDE IT IS ALSO A GLOBAL MIN/MAX

EX 1 FIND TWO NON-NEGATIVE NUMBERS WHOSE SUM IS 200 AND WHOSE PRODUCT IS MAXIMUM

STEP 1



STEP 2 MINIMIZE THE PRODUCT, xy

STEP 3 $P(x) = xy$

STEP 4 UGH, 2 VARIABLES, GET RID OF ONE USING A CONSTRAINT

THE SUM IS 200: $x + y = 200$

$y = 200 - x$

SUBSTITUTE: $P(x) = x(200 - x) = 200x - x^2$

STEP 5: x AND y NON-NEGATIVE: $x \in [0, 200]$, CLOSED INTERVAL

STEP 6: $P'(x) = 200 - 2x = 0$; $x = 100$

x	$P(x)$
0	0
100	10000
200	0

$x = 100$, $y = 100$ gives a max product of 10,000

CUT THIS SQUARE OUT

CUT THIS SQUARE OUT

x

$w = 8.5 - 2x$

FOLD

$l = 11 - 2x$

FOLD

FOLD

EX: THE CORNERS ARE CUT OUT OF AN $8\frac{1}{2}$" x 11" PIEC OF PAPER AND IT IS FOLDED INTO A BOX. WHAT SIZE SQUARES SHOULD BE REMOVED TO MAXIMIZE THE VOLUME?

STEP 1: LET x BE THE HEIGHT OF THE BOX

STEP 2: MAXIMIZE VOLUME

STEP 3: $V = l \cdot w \cdot h$

STEP 4:
$$V = (11-2x)(8.5-2x)x$$
$$= (93.5 - 39x + 4x^2)x$$
$$= 4x^3 - 39x^2 + 93.5x$$

STEP 5: x IS IN THE INTERVAL $[0, 4.25]$

STEP 6: $V'(x) = 12x^2 - 78x + 93.5 = 0$

TOO HARD TO FACTOR, USE QUADRATIC FORMULA!

$$x = \frac{78 \pm \sqrt{78^2 - 4 \cdot 12 \cdot 93.5}}{24}$$

$x \approx 1.585$ OR $\cancel{4.916}$ NOT IN DOMAIN.

x	$V(x)$
0	0
1.585	$\boxed{66.148 \text{ in}^3}$
4.25	0

GLOBAL MAX

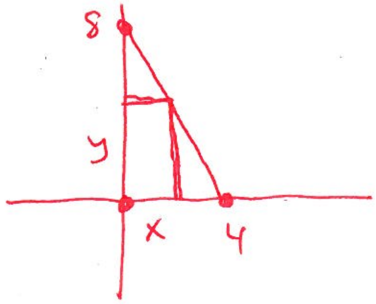
REMOVE SQUARES THAT ARE 1.585 x 1.585

FOLD

CUT THIS SQUARE OUT

CUT THIS SQUARE OUT

A rectangle is inscribed in the triangle with vertices (0,0), (4,0), (0,8), one side of the rectangle lying on the x-axis and one side lying on the y-axis. What is the maximum area of the rectangle?



Maximize area.

$A = xy$

constraint: upper right hand corner lies on line.

equation of line

$y = 8 - 2x$

substitute:

$A = x(8-2x) = 8x - 2x^2$

$A'(x) = 8 - 4x$

critical points:

$0 = 8 - 4x$

$8 = 4x$

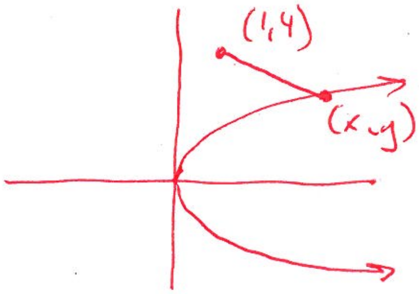
$x = 2$

domain: $x \in [0, 4]$

x	$A(x)$
0	0
2	$8 \cdot 2 - 2 \cdot 2^2 = 8$
4	0

Max area is 8.

Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1,4)$.



minimize distance

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

hint: minimize d^2

$$d^2 = (x-1)^2 + (y-4)^2$$

constraint: $y^2 = 2x$

$$f(y) = (x-1)^2 + (y-4)^2$$

$$= \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2$$

$$f'(y) = 2\left(\frac{y^2}{2} - 1\right)\cdot y + 2(y-4)$$

$$= (y^2 - 2)y + 2y - 8$$

$$= y^3 - 2y + 2y - 8$$

$$= y^3 - 8 = 0$$

$$y^3 = 8$$

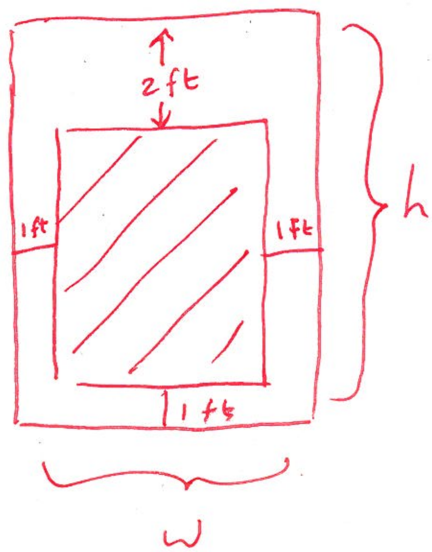
$$y = 2$$

domain $y \in (-\infty, \infty)$. open interval.

$f'(y)$: $-$ on $(-\infty, 2)$, $+$ on $(2, \infty)$

d^2 has a local minimum at $y = 2$, & there is only one c.p, so there is also a global min. there. So distance is minimum when $y = 2$. The closest point is $(2,2)$.

A rectangular mural will have a total area of 24 ft^2, which includes a border of 1 ft on the left, right and bottom, and a border of 2 ft. on the top. What dimensions maximize the total paintable area inside the borders?



constraint:

$$hw = 24\, ft^2$$

To maximize: area of shaded region.

$$A = (w-2)(h-3)$$

eliminate a variable using

$$hw = 24$$

$$h = \frac{24}{w}$$

substitute:

$$A = (w-2)\left(\frac{24}{w} - 3\right)$$

$$= 24 - \frac{48}{w} - 3w + 6$$

$$= 30 - \frac{48}{w} - 3w = 30 - 48w^{-1} - 3w$$

Differentiate:

$$A' = 48w^{-2} - 3$$

critical values:

$A' = 0 \Rightarrow \frac{48}{w^2} - 3 = 0 \Rightarrow \frac{48}{w^2} = 3 \Rightarrow w^2 = 16 \Rightarrow w = \pm 4$, but w is positive, so $w = 4$. Domain: $w \in [2, 8]$, a closed interval.

w	A
2	0 ft^2
4	6 ft^2
8	0 ft^2

Max area occurs when $w = 4\,ft$, $h = 6\,ft$

A can is to be ~~made~~ to hold 1 liter of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Minimize surface area

$A = 2\pi r^2 + 2\pi r h$

two variables, eliminate one using a constraint.

$V = 1000 \text{ cm}^3 = \pi r^2 h$

$h = \dfrac{1000}{\pi r^2}$

substitute:

$A = 2\pi r^2 + 2\pi r \cdot \dfrac{1000}{\pi r^2}$

$= 2\pi r^2 + \dfrac{2000 \cancel{\pi}}{\cancel{\pi} r} = 2\pi r^2 + \dfrac{2000}{r}$

Differentiate:

$A'(r) = 4\pi r - 2000 r^{-2}$

critical points:

$4\pi r - \dfrac{2000}{r^2} = 0$

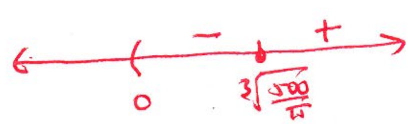
$4\pi r = \dfrac{2000}{r^2}$

$r^3 = \dfrac{500}{\pi}$

$r = \sqrt[3]{\dfrac{500}{\pi}}$

domain: $r \in (0, \infty)$

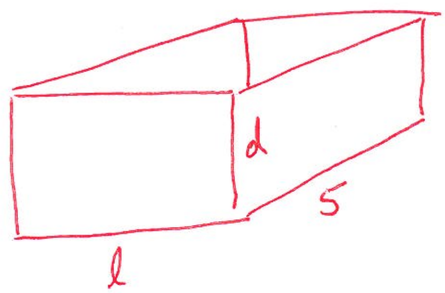
$A'(r)$



$r = \sqrt[3]{\frac{500}{\pi}}$ gives a local min for A. There is only one c.p., so there is a global min. at $r = \sqrt[3]{\frac{500}{\pi}}$

Dimensions: $r = \sqrt[3]{\dfrac{500}{\pi}}$, $h = \dfrac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2}$

A glass fish tank is to be constructed to hold 72 ft^3 of water. The top is to be open. Its width will be 5 ft but the length and depth are variable. Building the tank costs \$10 per square foot for the base and \$5 per square foot for the sides. What is the cost of the least expensive tank?



minimize cost.

$$C = \$5(2\cdot 5d + 2d\cdot l) + \$10\cdot 5l$$
$$= 50d + 10dl + 50l$$

constraint: volume $= 72\,ft^3$

$$5\cdot d\cdot l = 72$$
$$l = \frac{72}{5d}$$

Substituting:

$$C(d) = 50d + 10\cdot d\cdot \frac{72}{5d} + 50\cdot\frac{72}{5d}$$
$$= 50d + 144 + \frac{720}{d}$$

domain: d lies in the interval $(0, \infty)$

now minimize:

$$C'(d) = 50 - 720d^{-2} = 0$$
$$\frac{720}{d^2} = 50$$
$$d^2 = \frac{720}{50} = \frac{72}{5} \quad ; \quad d = \sqrt{\frac{72}{5}}$$

$C''(d) = 2\cdot 720d^{-3} > 0$.

By the 2nd derivative test, C has a local max at $d = \sqrt{\frac{72}{5}}$.

There is only one c.p., so C has a global max there.

$$C\left(\sqrt{\tfrac{72}{5}}\right) = 50\sqrt{\frac{72}{5}} + 144 + 720\sqrt{\frac{5}{72}} \approx \$523.47$$