1. A student throws a Frisbee across Norlin quad. The function $s(t)$ gives the distance in yards the Frisbee has flown after t seconds.

t in seconds	0	2	4	6	8	10	12	14	16
$s(t)$ in yards	0	15	28	39	48	55	60	63	64

(a) What is the average velocity of the Frisbee between $t = 2$ and $t = 10$ seconds? Include units.

We calculate average velocity by looking at the rate of change between two points $\left(\frac{\text{change in } y}{\text{change in } x}\right)$

$$\frac{s(10) - s(2)}{10 - 2} = \frac{55-15}{8} = \frac{40}{8} = \boxed{5 \text{ yards/sec}}$$

(b) Estimate the instantaneous velocity at $t = 14$ seconds. Include units.

Average Velocity on $[14, 16] \Rightarrow \frac{s(16) - s(14)}{16-14} = \frac{64-63}{2} = \frac{1}{2}$

Average Velocity on $[12, 14] \Rightarrow \frac{s(14) - s(12)}{14-12} = \frac{63-60}{2} = \frac{3}{2}$

Instantaneous Velocity at $t = 14$ is approximately the average of the average velocities $\Rightarrow \frac{\frac{1}{2} + \frac{3}{2}}{2} = \boxed{1 \text{ yard/sec}}$

(c) Assume that $s'(8) = 4$. What does the value 4 represent in the context of the problem? Include units.

4 is the instantaneous velocity at time 8 seconds; 4 is measured in units of yards/sec.

2. Evaluate the following limits. Show your work.

(a) $\lim_{x\to 0} \frac{e^{2x}}{\cos(2x)}$

Note both are continuous functions, so we can apply ~~directs~~ substitution property. Note also $\lim_{x\to 0} \cos(2x) \neq 0$, so we can use limit law $\frac{\lim_{x\to 0} e^{2x}}{\lim_{x\to 0}\cos(2x)}$ since both limits exist.

$$\boxed{\lim_{x\to 0} \frac{e^{2x}}{\cos(2x)} = \frac{1}{1} = 1}$$

(b) $\lim_{x\to 1} \frac{2-\sqrt{3+x}}{x-1}$

We can ~~use~~ conjugate to simplify.

$$\lim_{x\to 1} \frac{2-\sqrt{3+x}}{x-1}\left(\frac{2+\sqrt{3+x}}{2+\sqrt{3+x}}\right) = \lim_{x\to 1} \frac{4-3-x}{(x-1)(2+\sqrt{3+x})} = \lim_{x\to 1}\frac{(1-x)}{(2+\sqrt{3+x})(x-1)} = \lim_{x\to 1}\frac{-\cancel{(x-1)}}{(2+\sqrt{3+x})\cancel{(x-1)}}$$

Denominator limit does not zero after simplifying & both limit exist so we can use limit laws

$$\lim_{x\to 1} \frac{-1}{2+\sqrt{3+x}} = \boxed{\frac{-1}{4}}$$

(c) $\lim_{x\to 2} \frac{x^2-x-2}{x^2-2x}$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+1)}{x\cancel{(x-2)}} = \lim_{x\to 2}\frac{x+1}{x} = \boxed{\frac{3}{2}}$$

Again we can simplify and apply limit laws.

(d) $\lim_{x\to 3} \frac{|x-3|}{x^2-9}$

With absolute values we must check limits from both left and right.

For $x > 3$ (from right)

$|x-3| = x-3$ so $\lim_{x\to 3^+} \frac{|x-3|}{(x-3)(x+3)} = \lim_{x\to 3^+}\frac{\cancel{x-3}}{\cancel{(x-3)}(x+3)}$

$= \lim_{x\to 3^+} \frac{1}{x+3} = \frac{1}{6}$

For $x < 3$ (from left)

$|x-3| = -(x-3)$ so $\lim_{x\to 3^-} \frac{|x-3|}{(x-3)(x+3)} = \lim_{x\to 3^-}\frac{-\cancel{(x-3)}}{\cancel{(x-3)}(x+3)}$

$= \lim_{x\to 3^-} \frac{-1}{x+3} = -\frac{1}{6}$

Both left and right limits do not agree, so our limit does not exist.

3. Complete the definition of continuity.

A function f is continuous at a number a if:

$$\lim_{x \to a} f(x) = f(a)$$

This implies that for a function to be continuous ~~it~~ must satisfy the following

1) $\lim_{x \to a} f(x)$ must exist
2) The function must be defined at $x = a$
3) The limit of $f(x)$ as $x \to a$ must equal $f(a)$.

4. Consider the piece-wise function

$$f(x) = \begin{cases} e^{bc-3} & \text{,when } x < 1 \\ \ln(x) + 1 & \text{,when } x \geq 1 \end{cases}$$

Find the value of b that makes $f(x)$ continuous everywhere. Show your work.

e^{bx-3} is continuous everywhere

$\ln(x) + 1$ is continuous for $x \geq 1$

Find b so that $f(x)$ is continuous at $x = 1$

$$\lim_{x \to 1} f(x) = f(1) = \ln(1) + 1 = 0 + 1 = 1$$

So we must have $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 1$ to satisfy definition of continuity.

$$1 = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} e^{bx-3} = e^{b \cdot 1 - 3} = e^{b-3} \Rightarrow e^{b-3} = 1$$

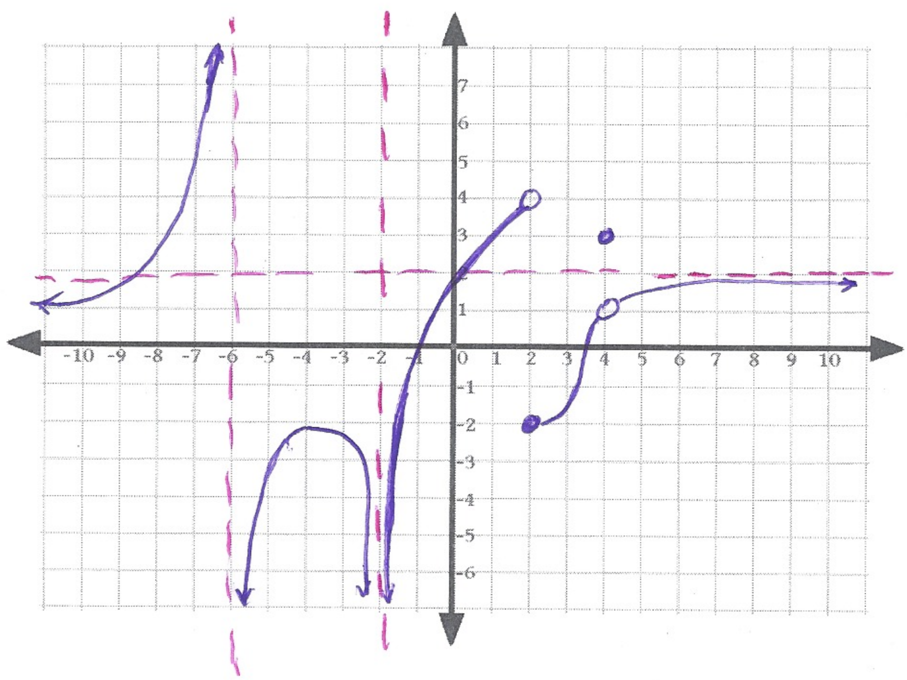
$$\Rightarrow b - 3 = 0$$

$$\Rightarrow \boxed{b = 3}$$

5. Sketch the graph of a function $f(x)$ which satisfies ALL the conditions below.

- f has an infinite discontinuity at $x = -6$
- $\lim_{x \to -3} f(x) = -\infty$
- $\lim_{x \to 2^-} f(x) = 4$
- $\lim_{x \to 2^+} f(x) = -2$
- $f(2) = -2$
- $\lim_{x \to 4} f(x) = 1$
- f has a removable discontinuity at $x = 4$
- $\lim_{x \to \infty} f(x) = 2$

There are many possible solutions, this is only one. As long as the conditions are satisfied, then your graph should be valid.



6. Evaluate the following limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{4t^2-3t+2}{t^4-2t^2+t-5} = 0$

Note degree of numerator is greater than degree of denominator

(b) $\displaystyle\lim_{x\to\infty} \frac{6x^3+x^2-4x+1}{3x^3-2x^2+5} = 2$

Note degree of numerator is equal to degree of denominator. Ratio of leading coefficients

$$\frac{6}{3} = 2$$

(c) $\displaystyle\lim_{x\to 1^+} 2^{3/(x-1)} = \infty$ As $x \to 1^+$: $\dfrac{3}{x-1} = \dfrac{3}{0^+} = \infty$,

then $2^{\frac{3}{x-1}} \to \infty$ as $x \to 1^+$.

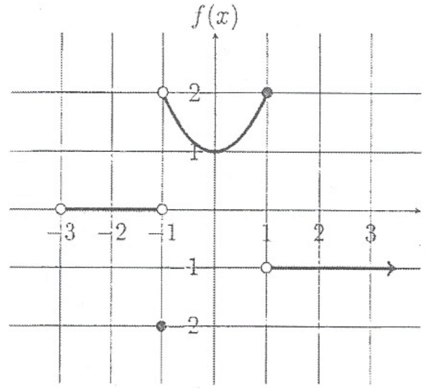
(d) $\displaystyle\lim_{x\to 2} \frac{x^2+2x-4}{x-2}$ Does not exist since if we observe $x^2+2x-4 \to 2^2+2\cdot 2+4 = 4$ as $x \to 2$

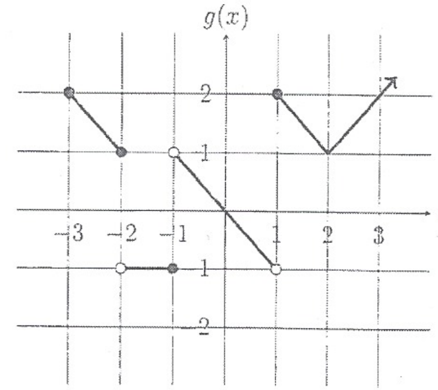
in numerator, but the denominator we see

$x-2 \to 0^+$ if $x \to 2^+$
$x-2 \to 0^-$ if $x \to 2^-$

Denominators are different, so as we approach from left + right we get different limit values.

7. The graphs of two piece-wise functions, $f(x)$ and $g(x)$, are shown below.





Evaluate the following limits.

(a) $\lim_{x\to 3} f(x)g(x)$ $= \left[\lim_{x\to3} f(x)\right]\left[\lim_{x\to3} g(x)\right] = (-1)(2) = -2$

(b) $\lim_{x\to 1} f(x)+g(x)$ $= \lim_{x\to1} f(x) + \lim_{x\to1} g(x) = -1 + 2 = 1$

(c) $\lim_{x\to 2^-} \dfrac{f(x)}{g(x)}$ $\doteq \dfrac{\lim_{x\to2^-} f(x)}{\lim_{x\to2^-} g(x)} = \dfrac{0}{1} = 0$

(d) $\lim_{x\to 2} f(g(x))$ $\stackrel{?}{=} 1$

$\lim_{x\to2^+} f(g(x))$: $g(x) \to 1^+$ as $x\to2^+$ and $f(x)\to -1$ as $x\to 1^+$

$\lim_{x\to2^-} f(g(x))$: $g(x) \to 1^+$ as $x\to2^-$ and $f(x)\to -1$ as $x\to 1^+$

limits agree, so our $\lim_{x\to2} f(g(x)) = -1$

8. Evaluate the following limit. Show all of your work. Be sure to cite any theorem(s) you use and justify why you can apply the theorem(s). $\lim_{x\to 3}(x-3)^2\cos(\frac{1}{x-3})$

Observe that $\cos\left(\frac{1}{x-3}\right)$ is bounded above by $+1$ and below by -1. Therefore, we can bound the function $(x-3)^2\cos\left(\frac{1}{x-3}\right)$ in the following way:

$$-(x-3)^2 \leq (x-3)^2\cos\left(\frac{1}{x-3}\right) \leq (x-3)^2$$

Then, if we apply the Squeeze Theorem we can take the limit of these functions as $x\to 3$

$$\lim_{x\to 3} -(x-3)^2 \leq \lim_{x\to 3}(x-3)^2\cos\left(\frac{1}{x-3}\right) \leq \lim_{x\to 3}(x-3)^2$$

$$0 \leq \lim_{x\to 3}(x-3)^2\cos\left(\frac{1}{x-3}\right) \leq 0$$

$$\Rightarrow \lim_{x\to 3}(x-3)^2\cos\left(\frac{1}{x-3}\right) = 0$$

9. Use the limit definition of derivative to compute

$$f'(1) \text{ if } f(x) = x^2 + x.$$

Recall the definition of derivative at a point a is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

So $$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{((1+h)^2 + (1+h)) - (1^2+1)}{h}$$

$$= \lim_{h \to 0} \frac{1 + 2h + h^2 + 1 + h - 2}{h}$$

$$= \lim_{h \to 0} \frac{3h + h^2}{h} = \lim_{h \to 0} 3 + h = 3 + 0 = \boxed{3}$$

10. Use the Intermediate Value Theorem to show that the equation

$$x^3 + x^2 + x - 2 = 0$$

has a solution in the interval [0,1]. You must check that the hypotheses (conditions) of the Intermediate Value Theorem are satisfied before you may apply it.

To apply IVT we must verify the function
① $f(x) = x^3 + x^2 + x - 2$ is continuous,
② that $f(0) < f(1)$ or $f(1) < f(0)$, and
③ that 0 is between (or lies in) the closed interval $[0,1]$.

1) Note $f(x)$ is a polynomial $\Rightarrow$ continuous function since the limit as x approaches any value will equal the function evaluated at that point following limit laws.

2) $f(0) = -2$ and $f(1) = 1 \Rightarrow f(0) < f(1)$

3) 0 lies in the closed interval $[0,1]$, in fact it is the endpoint of the interval.

Since these conditions/hypotheses are satisfied we can apply the IVT which allows us to conclude that there exists a number c in the closed interval $[0,1]$ such that
$f(c) = 0$.