Calculus 1: A Large and In Charge Review

1. Calculate the average rate of change for the function over the given interval.

   (a) $f(x) = x^2 + 1$, $[-1, 2]$

   (b) $g(x) = \sqrt{x}$, $[4, 25]$

   (c) $h(x) = \frac{1}{x^2+1}$, $[-1, 3]$

2. Calculate the following limits, if it exists.

   (a) $\lim_{x\to 12} 10 - 3x$

   (b) $\lim_{x\to 5} \dfrac{4}{x-7}$

   (c) $\lim_{x\to -3} \dfrac{x^2 - x - 12}{x+3}$

   (d) $\lim_{x\to 1} \dfrac{x^2 + x - 2}{x^2 - 3x + 2}$

   (e) $\lim_{x\to 1} \dfrac{x^3 - 1}{x^2 - 1}$

   (f) $\lim_{x\to 1} \dfrac{\sqrt{x} - x^2}{1 - \sqrt{x}}$

   (g) $\lim_{x\to -4^-} \dfrac{|x+4|}{x+4}$

   (h) $\lim_{x\to 1.5} \dfrac{2x^2 - 3x}{|2x-3|}$

   (i) $\lim_{x\to 0^+} \dfrac{1}{x} - \dfrac{1}{|x|}$

3. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous on $\mathbb{R}$.

   (a) $f(x) = \dfrac{x^2 - 2x - 8}{x+2}$, $a = -2$

   (b) $f(x) = \dfrac{x-7}{|x-7|}$, $a = 7$

   (c) $f(x) = \dfrac{x^3 + 64}{x+4}$, $a = -4$

4. Find and prove the following limits.

   (a) $\lim_{x\to -1} f(x)$ if $1 \leq f(x) \leq x^2 + 2x + 2$

   (b) $\lim_{x\to 1} f(x)$ if $3x \leq f(x) \leq x^3 + 2$

   (c) $\lim_{x\to 0} x^3 \cos\left(\dfrac{1}{x}\right)$

   (d) $\lim_{x\to 0^+} \sqrt{x}\left[1 + \sin^2(2\pi/x)\right]$

5. Prove there exists a solution to the given equations on the given intervals.

   (a) $x^2 = \sqrt{x+1}$, $(1, 2)$

   (b) $\cos x = 2x$, $(0, \frac{\pi}{4})$

6. Using the definition of the derivative, calculate the following derivatives.

   (a) $y = 3x + 1$

   (b) $y = \sqrt{x}$

   (c) $y = x^3 + x + 1$

   (d) $y = \dfrac{1}{\sqrt{x}}$

7. Find the values of x where the functions are **1. continuous** and **2. differentiable**.

   (a) $f(x) = 3x^2 + 1$

   (b) $f(x) = x + \sqrt{x}$

   (c) $f(x) = \dfrac{x+1}{x-1}$

   (d) $f(x) = \dfrac{1}{x^2+1} + \sqrt{x-2} + |x|$

8. Differentiate the following functions.

   (a) $\frac{x+1}{x-1}$

   (b) $y = \frac{1-u^2}{1+u^2}$

   (c) $y = \frac{x^2+4x+3}{\sqrt{x}}$

   (d) $y = \frac{\sqrt{x}-1}{\sqrt{x}+1}$

   (e) $y = ax^2 + bx + c$

   (f) $g(x) = x + \sqrt[5]{x^2}$

   (g) $u = \sqrt[3]{t^2} + 2\sqrt{t^3}$

   (h) $s = \sqrt{t}(t^3 - \sqrt{t} + 1)$

9. Find an equation of the tangent line to the curve at the given point.

   (a) $y = \frac{2x}{x+1}$, $(1, 1)$

   (b) $y = \frac{\sqrt{x}}{x+1}$, $x = 4$

   (c) $y = x + \sqrt{x}$ $x = 1$

10. Given that $f(1) = 2$, $f(2) = 5$, $f'(1) = -1$, $f'(2) = 0$ and $g(1) = 1$, $g(2) = 2$, $g'(1) = 3$, $g'(2) = 4$, evaluate the following:

    (a) $(f + g)'(1)$

    (b) $(2f - g)'(2)$

    (c) $(3fg)'(1)$

    (d) $\left(\frac{f}{g}\right)'(1)$

    (e) $(f \circ g)'(1)$

    (f) $(f^2 \cdot g)'(1)$

    (g) $(\sqrt{fg})'(2)$

11. For the following position functions, find the position, velocity, speed and acceleration at time $t = 1$ and $t = 4$

    (a) $s(t) = 3t + 2$

    (b) $s(t) = 3t^3 - 2t + 1$

    (c) $s(t) = -3t^2 + 16t + 92$

    (d) $s(t) = \frac{1}{1+t^2}$

12. Differentiate.

(a) $f(x) = x \sin x$ (d) $h(\theta) = \sqrt{\theta} \cot \theta$ (g) $y = x \sin x \cos x$

(b) $y = \cos x - 2 \tan x$ (e) $y = \frac{\sin x}{1+\cos x}$

(c) $g(t) = 4 \sec t + 2 \tan t$ (f) $y = \tan \theta(\sin \theta + \cos \theta)$ (h) $y = \csc x \cot x$

13. Find the slope of the tangent line at $x = 0$

(a) $y = (x^2 - x + 1)^3$ (d) $y = a^3 + \cos^3 x$ (g) $y = \sin(\sin(\sin x))$

(b) $y = \frac{1}{(x^2-2x-5)^4}$ (e) $y = (x^2 + 1)\sqrt[3]{x^2 + 2}$ (h) $y = \sqrt{\cos(\sin^2 x)}$

(c) $y = \sqrt[3]{1 + \tan t}$ (f) $y = \tan(\cos x)$

14. Find $\frac{dy}{dx}$

(a) $x^2 - y^2 = 1$

(b) $x^2 - 2xy + y^3 = c$

(c) $\sqrt{x + y} + \sqrt{xy} = 6$

15. Find the line tangent to the curve at the given point.

(a) $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, $(3, 1)$

(b) $x^2y^2 = (y + 1)^2(4 - y^2)$, $(0, -2)$

(c) $y^2 = 5x^4 - x^2$, $(1, 2)$

16. Assuming $g(x)$ is twice differentiable, find f'' in terms of g, g' and g''.

(a) $f(x) = xg(x^2)$

(b) $f(x) = \frac{g(x)}{x}$

(c) $f(x) = g(\sqrt{x})$

17. If V is the volume of a cube with edge length x and the cube expands as time passes, find $\frac{dV}{dt}$ in terms of $\frac{dx}{dt}$

18. A particle moves along the curve $y = \sqrt{1 + x^2}$. As it reaches the point $(2, 3)$, the y-coordinate is increasing at a rate of 4 cm/s. How fast is the x-coordinate of the point changing at that instant?

19. At noon, ship A is 150 km west of ship B is sailing east at 35 km/h and ship B is sailing north at 25km/h. How fast is the distance between the ships changing at 4 PM?

20. Water is leaking out of an inverted conical tank at a rate of 10,000 cubic cm per minute at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

21. Two sides of a triangle have lengths 4 m and 5 m. The angle between them is increasing at a rate of 0.06 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of the fixed length is $\pi/3$.

22. A runner sprints around a circular track of radius 100m at a constant speed of 7 m/s. The runner's friend is standing at a distance of 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?

23. Using linear approximation (or differentials) to estimate the following.

    (a) $\sqrt{36.1}$
    (b) $\frac{1}{10.1}$
    (c) $(1.97)^6$

24. Find the critical values of the function.

    (a) $f(x) = 5x^2 + 4x$
    (b) $f(t) = 2t^3 + 3t^2 - 6t + 4$
    (c) $s(t) = t^4 + 4t^3 + 2t^2$
    (d) $f(r) = \frac{r}{r^2+1}$
    (e) $g(x) = x^{1/3} - x^{-2/3}$
    (f) $g(x) = \sqrt[3]{x^2 - x}$

25. Find the absolute maximum and absolute minimum values of f on the given interval.

    (a) $f(x) = 3x^2 - 12x + 5,\ [0, 3]$
    (b) $f(x) = 2x^3 + 3x^2 + 4,\ [-2, 1]$
    (c) $f(x) = x^2 + \frac{2}{x},\ [1/2, 2]$
    (d) $f(x) = \frac{x}{x^2+1},\ [0, 2]$
    (e) $f(x) = \sin x + \cos x,\ [0, \pi/4]$
    (f) $f(x) = x - 2\cos x,\ [-\pi, \pi]$

26. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

    (a) $f(x) = 3x^2 + 2x + 5,\ [-1, 1]$
    (b) $f(x) = x^3 + x - 1,\ [0, 2]$
    (c) $f(x) = \sqrt[3]{x},\ [0, 1]$
    (d) $f(x) = \frac{x}{x+2},\ [1, 4]$

27. Show that $x^5 + 10x + 3 = 0$ has exactly one real root.

28. Show there does not exist a function f such that $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$ for all x.

29. For each function, **1** Find the intervals of increase and decrease, **2** Find the local maximum and minimum values, **3** Find the intervals of concavity and inflection points and **4** Sketch the graph.

   (a) $f(x) = 2x^3 - 3x^2 - 12x$
   (b) $f(x) = x^4 - 6x^2$
   (c) $h(x) = (x^2 - 1)^3$
   (d) $P(x) = x\sqrt{x^2 + 1}$
   (e) $Q(x) = x\sqrt{x + 1}$
   (f) $f(x) = x - 3x^{1/3}$
   (g) $f(t) = t + \cos t$, $-2\pi \leq t \leq 2\pi$

30. Evaluate the following limits.

   (a) $\displaystyle\lim_{x\to\infty} \frac{6x^2 + 5x}{(1 - x)(2x - 3)}$
   (b) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2 + 4x}}{4x + 1}$
   (c) $\displaystyle\lim_{x\to\infty} \sqrt{x^2 + 3x + 1} - x$
   (d) $\displaystyle\lim_{x\to\infty} \frac{1 - \sqrt{x}}{1 + \sqrt{x}}$
   (e) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{4x^2 + 1}}{x + 1}$

31. Find two positive numbers whose product is 100 and whose sum is a minimum.

32. A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can she do this so as to minimize the cost of the fence?

33. A rectangular storage container with an open top is to have a volume of 10 cubic meters. The length of its base is twice the width. Material for the base costs \$10 per square meter and \$6 per square meter for the sides. Find the minimal cost of such a box.

34. For each function, find the general antiderivative.

   (a) $f(x) = 6x^2 - 8x + 3$
   (b) $f(x) = 1 - x^3 + 5x^5 - 3x^7$
   (c) $f(x) = 5x^{1/4} - 7x^{3/4}$
   (d) $f(x) = \frac{10}{x^9}$
   (e) $f(x) = \frac{3}{x^2} - \frac{5}{x^4}$

35. Find f.

(a) $f''(x) = 6x + 12x^2$

(b) $f''(x) = 1 + x^{4/5}$

(c) $f'''(t) = 60t^2$

(d) $f'''(t) = t - \sqrt{t}$

(e) $f'(x) = 1 + 1/x^2,\ f(1) = 2$

(f) $f'(x) = 3\cos x + 5\sin x,\ f(0) = 4$

(g) $f''(x) = 12x^2 - 6x + x,\ f(0) = 1,\ f(2) = 11$

36. Express the area under the curve as Riemann Sum (The number of rectangles goes to infinity). Do not evaluate the sum.

(a) $y = \sqrt[3]{x},\ 0 \le x \le 8$

(b) $f(x) = x + \sin x,\ \pi \le x \le 2\pi$

37. Estimate the area under the curve with **1.** $n = 2$, **2.** $n = 4$ and **3.** $n = \infty$.

(a) $\displaystyle\int_{-1}^{3} (1 + 3x)\ dx$

(b) $\displaystyle\int_{0}^{2} (2 - x^2)\ dx$

(c) $\displaystyle\int_{0}^{4} (1 + 2x^3)\ dx$

38. Evaluate the integral by interpreting it in terms of area.

(a) $\displaystyle\int_{1}^{3} (1 + 2x)\ dx$

(b) $\displaystyle\int_{-3}^{0} (1 + \sqrt{9 - x^2})\ dx$

(c) $\displaystyle\int_{-2}^{2} (1 - |x|)\ dx$

39. Find the derivative of each of the following functions.

(a) $g(x) = \displaystyle\int_{0}^{x} \sqrt{1 + 2t}\ dt$

(b) $g(y) = \displaystyle\int_{2}^{y} t^2 \sin t\ dt$

(c) $F(x) = \displaystyle\int_{x}^{2} \cos(t^2)\ dt$

(d) $\displaystyle\int_{x}^{10} \tan\theta\ d\theta$

(e) $\displaystyle\int_{0}^{x^2} \sqrt{1 + r^2}\ dr$

(f) $\displaystyle\int_{0}^{3+x^3} \sin t\ dt$

40. Evaluate the integral.

(a) $\displaystyle\int_1^2 (5x^2 - 4x + 3)\,dx$

(b) $\displaystyle\int_0^1 (y^9 - 2y^5 + 3y)\,dy$

(c) $\displaystyle\int_1^2 \frac{t^6 - t^2}{t^4}\,dt$

(d) $\displaystyle\int_0^2 (x^3 - 1)^2\,dx$

(e) $\displaystyle\int_1^{-1} (x-1)(3x+2)\,dx$

(f) $\displaystyle\int_{\pi/3}^{\pi/2} \csc x \cot x\,dx$

41. Find the **total** area bounded by the curve.

(a) $y = 3x - 1$, $-2 \le x \le 4$

(b) $y = x^2 - x - 2$, $-3 \le x \le 3$

42. Evaluate the following indefinite integrals. Don't forget the "$+C$"!

(a) $\displaystyle\int x^3(1-x^4)^5\,dx$

(b) $\displaystyle\int (2-x)^6\,dx$

(c) $\displaystyle\int x(x^2+1)^{3/2}\,dx$

(d) $\displaystyle\int \frac{1}{(1-3x)^4}\,dx$

(e) $\displaystyle\int \sqrt[5]{3-5x}\,dx$

(f) $\displaystyle\int \sec^2 3x\,dx$

(g) $\displaystyle\int \frac{(1+\sqrt{x})^9}{\sqrt{x}}\,dx$

(h) $\displaystyle\int \frac{\cos(\pi/x)}{x^2}\,dx$

(i) $\displaystyle\int \sec^3 x \tan x\,dx$

43. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^7 \sqrt{4+3x}\,dx$

(b) $\displaystyle\int_0^{\sqrt{\pi}} x\cos x^2\,dx$

(c) $\displaystyle\int_0^{\pi/4} \sin 4t\,dt$

(d) $\displaystyle\int_1^4 \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\,dx$

(e) $\displaystyle\int_0^1 \cos \pi t\,dt$

(f) $\displaystyle\int_0^a x\sqrt{a^2-x^2}\,dx$

44. Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x^2\sin x}{1+x^6}\,dx$

45. Evaluate $\displaystyle\int_0^1 x\sqrt{1-x^4}\,dx$.

46. If f is continuous and $\displaystyle\int_0^9 f(x)\,dx = 4$, find $\displaystyle\int_0^3 xf(x^2)\,dx$.

47. Find $\lim_{h\to 0} \frac{1}{h}\int_2^{2+h} \sqrt{1+t^3}\,dt$.
Hint: Its not zero.

48. Have a nice break!