Calculus 1: A Large and In Charge Review – Solutions

I use the symbol ∃ which is shorthand for the phrase "there exists".

1. We use the formula that Average Rate of Change is given by $\frac{f(b)-f(a)}{b-a}$

   (a) $\frac{5-2}{2-(-1)} = \frac{3}{3} = 1$

   (b) $\frac{5-2}{25-4} = \frac{3}{21} = \frac{1}{7}$

   (c) $\frac{1/10-1/2}{3-(-1)} = \frac{-4/10}{4} = -\frac{1}{10}$

2. (a) $\lim_{x\to 12} 10 - 3x = 10 - 26 = -26$

   (b) $\lim_{x\to 5} \frac{4}{x-7} = \frac{4}{-2} = -2$

   (c) $\lim_{x\to -3} \frac{x^2-x-12}{x+3} = \lim_{x\to -3} \frac{(x-4)(x+3)}{x+3} = -7$

   (d) $\lim_{x\to 1} \frac{x^2+x-2}{x^2-3x+2} = \lim_{x\to 1} \frac{(x+2)(x-1)}{(x-1)(x-2)} = \frac{3}{-1} = -3$

   (e) $\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \frac{3}{2}$

   (f) $\lim_{x\to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}} = \lim_{x\to 1} \frac{\sqrt{x}(1-x^{3/2})}{1-\sqrt{x}} = \lim_{x\to 1} \frac{\sqrt{x}(1-\sqrt{x})(1+\sqrt{x}+x)}{1-\sqrt{x}} = 3$

   (g) $\lim_{x\to -4^-} \frac{|x+4|}{x+4} = \lim_{x\to -4^-} \frac{-(x+4)}{x+4} = -1$

   (h) $\lim_{x\to 1.5} \frac{2x^2-3x}{|2x-3|} = \lim_{x\to 1.5} \frac{x(2x-3)}{|2x-3|}$ does not exist. (One sided limits do not agree).

   (i) $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{|x|} = \lim_{x\to 0^+} \frac{1}{x} - \frac{1}{x} = 0$

3. (a) $\lim_{x\to -2} \frac{x^2-2x-8}{x+2} = \lim_{x\to -2} \frac{(x-4)(x+2)}{x+2} = \lim_{x\to -2} x-4 = -6$ So set $g(x) = x-4$ and $g(x)$ is a continuous extension of $f(x)$.

   (b) This is impossible as $\lim_{x\to 7} \frac{x-7}{|x-7|}$ does not exist. Indeed since $\lim_{x\to 7^+} \frac{x-7}{|x-7|} = \lim_{x\to 7^+} \frac{x-7}{x-7} = 1$ and $\lim_{x\to 7^-} \frac{x-7}{|x-7|} = \lim_{x\to 7^-} \frac{x-7}{-(x-7)} = -1$

(c) $\lim_{x\to -4} \dfrac{x^3+64}{x+4} = \lim_{x\to -4} \dfrac{(x+4)(x^2-4x+16)}{x+4} = 48$ So set $g(x) = x^2-4x+16$ and $g(x)$ is a continuous extensions of $f(x)$.

4. (a) By the Sandwich Theorem, $\lim_{x\to -1} f(x) = 1$

   (b) By the Sandwich Theorem, $\lim_{x\to 1} f(x) = 3$

   (c) Observe $-1 \le \cos\left(\frac{1}{x}\right) \le 1 \Rightarrow -x \le x\cos\left(\frac{1}{x}\right) \le x$ for $x > 0$ and $x \le x\cos\left(\frac{1}{x}\right) \le -x$ for $x < 0$. So by the Sandwich Theorem, each directional limit is 0 and so $\lim_{x\to 0} x\cos\left(\dfrac{1}{x}\right) = 0$

   (d) Note that $1 \le 1+\sin^2\left(\frac{2\pi}{x}\right) \le 2 \Rightarrow \sqrt{x} \le \sqrt{x}\left(1+\sin^2\left(\frac{2\pi}{x}\right)\right) \le 2\sqrt{x}$ as $\sqrt{x} > 0$ for so by the Sandwich Theorem $\lim_{x\to 0^+} \sqrt{x}\left(1+\sin^2\left(\dfrac{2\pi}{x}\right)\right) = 0$

5. (a) Consider $x^2 - \sqrt{x+1} = f(x)$. $f(x)$ is continuous on $(1,2)$ as it is the difference of continuous functions and so $f(1) = 1-\sqrt{2} < 0$ and $f(2) = 4-\sqrt{3} > 0$. By the Intermediate Value Theorem, $\exists\ c$ such that $f(c) = 0$ with $1 < c < 2$, i.e. $(c)^2 = \sqrt{c+1}$

   (b) Again, $f(x) = \cos x - 2x$ is continuous on $(0, \pi/4)$ and $f(0) = 1 > 0$ and $f(\pi/4) < 0$ so by the IVT, a solution exists.

6. Use the definition $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

   (a) $\lim_{h\to 0} \dfrac{3(x+h)+1-(3x+1)}{h} = \lim_{h\to 0} \dfrac{3x+3h+1-3x-1}{h} = \lim_{h\to 0} \dfrac{3h}{h} = 3$

   (b) $\lim_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \dfrac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \dfrac{1}{\sqrt{x+h}+\sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

   (c) $\lim_{h\to 0} \dfrac{(x+h)^3+(x+h)+1-(x^3+x+1)}{h}$

   $= \lim_{h\to 0} \dfrac{x^3+3x^2h+3xh^2+h^3+x+h+1-x^3-x-1}{h}$

   $= \lim_{h\to 0} = \dfrac{h(3x^2+3xh+h^2+1}{h} = 3x^2+1$

   (d) $\lim_{h\to 0} \dfrac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \dfrac{\sqrt{x}-\sqrt{x+h}}{h(\sqrt{x}\sqrt{x+h})} = \lim_{h\to 0} \dfrac{x-x-h}{h(\sqrt{x}\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}$

   $= \lim_{h\to 0} \dfrac{-1}{h(\sqrt{x}\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})} = \dfrac{-1}{x(2\sqrt{x})} = -\dfrac{1}{2x^{3/2}}$

7. $f(x)$ being differentiable at a point means that $f'(x)$ exists and is continuous at that point.

   (a) $f'(x) = 6x$ so both f and $f'(x)$ are continuous and differentiable everywhere.

   (b) $f(x)$ is continuous for $x \geq 0$ and $f'(x) = 1 + \frac{1}{2\sqrt{x}}$ so $f(x)$ is differentiable for $x > 0$

   (c) $f(x)$ is continuous for any $x \neq 1$ and $f'(x) = \frac{-2}{(x-1)^2}$ so $f(x)$ is differentiable for any $x \neq 1$

   (d) $f(x)$ is continuous for $x \geq 2$ and $f'(x)$ is continuous on $(0, 2)$ and $(2, \infty)$.

8. (a) $f'(x) = \frac{(1)(x-1)-(1)(x+1)}{(x-1)^2} = \frac{x-1-x-1}{(x-1)^2} = \frac{-2}{(x-1)^2}$

   (b) $\frac{dy}{du} = \frac{-2u(1+u^2)-2u(1-u^2)}{(1+u^2)} = \frac{-2u-2u^3-2u+2u^3}{(1+u^2)} = \frac{-4u}{(1+u^2)^2}$

   (c) $y = x^{3/2} + 4\sqrt{x} + 3x^{-1/2} \Rightarrow y' = \frac{3}{2}\sqrt{x} + \frac{2}{\sqrt{x}} - \frac{3}{2x^{3/2}}$

   (d) $y = \frac{\sqrt{x}-1}{\sqrt{x}+1} \Rightarrow y' = \frac{\frac{1}{2\sqrt{x}}(\sqrt{x}+1) - \frac{1}{2\sqrt{x}}(\sqrt{x}-1)}{(\sqrt{x}+1)^2} = \frac{1}{2\sqrt{x}(\sqrt{x}+1)^2}$

   (e) $y' = 2ax + b$

   (f) $g(x) = x + 2^{2/5} \Rightarrow g'(x) = 1 + \frac{2}{5}x^{-3/5}$

   (g) $u = t^{2/3} + 2t^{1/3} \Rightarrow \frac{du}{dt} = \frac{2}{3}t^{-1/3} + \frac{2}{3}t^{-2/3}$

   (h) $s = t^{7/2} - t + \sqrt{t} \Rightarrow \frac{ds}{dt} = \frac{7}{2}t^{5/2} - 1 + \frac{1}{2\sqrt{t}}$

9. Use that the tangent line at a point $x = c$ is given by $y - y_0 = f'(x)(x - c)$

   (a) $y' = \frac{2(x+1)-2x(1)}{(x+1)^2} \Rightarrow m = \frac{2}{4} = \frac{1}{2}$.
   So the tangent line is $y - 1 = \frac{1}{2}(x - 1)$.

   (b) $y' = \frac{\frac{1}{2\sqrt{x}}(x+1)-\sqrt{x}}{(x+1)^2} \Rightarrow m = \frac{\frac{1}{4}(5)-2}{25} = \frac{-3/4}{25} = -\frac{3}{100}$.
   So the tangent line is $y - \frac{2}{5} = -\frac{3}{100}(x - 4)$.

   (c) $y' = 1 + \frac{1}{2\sqrt{x}} \Rightarrow m = \frac{3}{2}$.
   So the tangent line is $y - 2 = \frac{3}{2}(x - 1)$.

10. Using the rules of derivatives, it is quite simple.

   (a) $(f + g)'(1) = f'(1) + g'(1) = 2$

   (b) $(2f - g)'(2) = 2f'(2) - g'(2) = -4$

   (c) $(3fg)'(1) = 2f'(1)g(1) + 3f(1)g'(1) = 15$

   (d) $\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1)-f(1)g'(1)}{(g(1))^2} = -7$

(e) $(f \circ g)'(1) = f'(g(1))g'(1) = f'(1)g'(1) = -3$

(f) $(f^2 \cdot g)'(1) = 2f(1)f'(1)g(1) + (f(1))^2 g'(1) = 8$ (Chain/Product Rule)

(g) $(\sqrt{fg})'(2) = \frac{1}{2\sqrt{f(2)g(2)}} \cdot (fg)'(2) = \frac{1}{2\sqrt{f(2)g(2)}}(f'(2)g(2) + f(2)g'(2)) = \sqrt{10}$

11. Use that $\frac{d}{dt}(s(t)) = v(t)$, $\frac{d}{dt}(v(t)) = a(t)$ and speed is the absolute value of velocity. The answers here are given so at the first number is velocity, speed then acceleration for $t = 1$, given as $\big(v(1), |v(1)|, a(1)\big)$ and the same for time $t = 4$.

(a) $v(t) = 3$ and $a(t) = 0$ so $\Rightarrow (3, 3, 0)$ and $(3, 3, 0)$

(b) $v(t) = 9t^2 - 2$ and $a(t) = 18t$ so $\Rightarrow (7, 7, 18)$ and $(142, 288, 0)$

(c) $v(t) = -6t + 16$ and $a(t) = -6$ so $\Rightarrow (10, 10, -6)$ and $(-8, 8, 0)$

(d) $v(t) = \frac{-2t}{(1+t^2)^2}$ and $a(t) = \frac{-2(1+t^2)^2+2t(2(1+t^2)(2t))}{(1+t^2)^4}$ so $\Rightarrow (-1, 1, \frac{1}{2})$ and $(\frac{-4}{289}, \frac{4}{289}, \frac{94}{4913})$

12. (a) $f'(x) = \sin x + x\cos x$

(b) $\frac{dy}{dx} = -\sin x - 2\sec^2 x$

(c) $g'(t) = 4\sec t\tan t + 2\sec^2 t$

(d) $h'(\theta) = \frac{1}{2\sqrt{\theta}}\cot\theta - \sqrt{\theta}\csc^2\theta$

(e) $\frac{dy}{dx} = \frac{\cos x(1+\cos x)-\sin x(-\sin x)}{(1+\cos x)^2} = \frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2} = \frac{\cos x+1}{(1+\cos x)^2} = \frac{1}{1+\cos x}$

(f) $y' = \sec^2(\cos x)(-\sin x)$

(g) $y' = \sin x\cos x + x\cos^2 x - x\sin^2 x$

(h) $y' = -\csc x\cot^2 x - \csc^3 x$

13. This is the same as evaluating the derivative at $x = 0$.

(a) $y' = 3(x^2 - x + 1)^2(2x - 1) \Rightarrow m = -3$

(b) $y' = -4(x^2 - 2x - 5)^{-5}(2x - 2) \Rightarrow m = \frac{-8}{3125}$

(c) $y' = \frac{1}{3}(1 + \tan t)^{-2/3} \Rightarrow m = \frac{1}{3}$

(d) $y' = 3\cos^2 x(-\sin x) \Rightarrow m = 0$

(e) $y' = 2x\sqrt[3]{x^2 + 2} + (x^2 + 1)\frac{1}{3}(x^2 + 2)^{-2/3}(2x) \Rightarrow m = 0$

(f) $y' = \sec^2(\cos x)(-\sin x) \Rightarrow m = 0$

(g) $y' = \cos(\sin(\sin x))\cos(\sin(x))\cos x \Rightarrow m = 1$

(h) $y' = \frac{1}{2\sqrt{\cos(\sin^2 x)}}(-\sin(\sin^2 x))2\sin x\cos x \Rightarrow m = 0$

14. Use the method of implicit differentiation.

(a) $\Rightarrow 2x - 2x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

(b) $\Rightarrow 2x - 2y - 2x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{2y-2x}{3y^2-2x}$

(c) $\Rightarrow \frac{1}{2\sqrt{x+y}}\left(1+\frac{dy}{dx}\right) + \frac{1}{2\sqrt{xy}}\left(y+x\frac{dy}{dx}\right) = 0 \Rightarrow \frac{dy}{dx} = \frac{-\frac{\sqrt{y}}{2\sqrt{x}}-\frac{1}{2\sqrt{x+y}}}{\frac{\sqrt{x}}{2\sqrt{y}}+\frac{1}{2\sqrt{x+y}}} = -\frac{y\sqrt{x}\sqrt{x+y}+(x+y)\sqrt{x}\sqrt{y}}{x\sqrt{y}\sqrt{x+y}+(x+y)\sqrt{x}\sqrt{y}}$

15. This is a combination of problem 10 and 15.

(a) $\Rightarrow 4(x^2+y^2)(2x+2y\frac{dy}{dx}) = 25(2x-2y\frac{dy}{dx}) \Rightarrow 4(10)(6+2m) = 25(6-2m) \Rightarrow m = \frac{-9}{13}$
So the tangent line is $y - 1 = \frac{-9}{13}(x-3)$

(b) $\Rightarrow 2xy^2 + 2x^2y\frac{dy}{dx} = 2(y+1)(4-y^2)\frac{dy}{dx} + (y+1)^2(-2y)\frac{dy}{dx} \Rightarrow m = 0$
So the tangent line is $y = -2$

(c) $\Rightarrow 2y\frac{dy}{dx} = 20x^3 - 2x \Rightarrow 4m = 18 \Rightarrow m = \frac{9}{2}$
So the tangent line is $y - 2 = \frac{9}{2}(x-1)$

16. Just use the rules of taking derivatives.

(a) $f'(x) = g(x^2) + xg'(x^2)(2x) = g(x^2) + 2x^2g'(x^2)$
$\Rightarrow f''(x) = 2xg'(x^2) + 4xg'(x^2) + 8x^3g''(x^2) = 6xg'(x^2) + 8x^3g''(x^2)$

(b) $f'(x) = \frac{g'(x)x-g(x)}{x^2} = \frac{g'(x)}{x} - \frac{g(x)}{x^2} \Rightarrow f''(x) = \frac{g''(x)}{x} - \frac{2g'(x)}{x^2} + \frac{2g(x)}{x^3}$

(c) $f'(x) = g'(\sqrt{x})\frac{1}{2\sqrt{x}} \Rightarrow f''(x) = g''(x)\frac{1}{4x} - g'(x)\frac{1}{4x^{3/2}}$

17. $V = x^3$ so $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$

18. $\frac{dy}{dt} = \frac{x}{\sqrt{1+x^2}}\frac{dx}{dt} \Rightarrow 4 = \frac{2}{\sqrt{5}}\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2\sqrt{5}$

19. Let s be the distance between the ships so $s^2 = x^2 + y^2$ where x is the distance from Ship A to where Ship B was at noon and Y is the same for Ship B. The at the time in question, $s = \sqrt{100 + 10,000} = 10\sqrt{101}$. So:

$$\Rightarrow s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} \Rightarrow 10\sqrt{101}\frac{ds}{dt} = (10)(-35) + 100(25) \Rightarrow \frac{ds}{dt} = \frac{215}{\sqrt{101}}$$

20. Let r be the radius of the water level inside the cone and h its height. By similar triangles, we deduce that $h = 3r$. Note we must use the same units for all measurement so I converted everything into centimeters. Thus the final answer is in cubic centimeters per minute. Thus:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{27}\pi h^3 \Rightarrow \frac{dV}{dt} = \frac{1}{9}\pi h^2\frac{dh}{dt} \Rightarrow \frac{dV}{dt} = \frac{1}{9}\pi(200)^2(20) = \frac{800,000\pi}{9}$$

Since this is the *overall* change in volume, the rate at which water is being pumped into the tank is $\frac{800,000\pi}{9} + 10,000$ since the rate of change is the rate in minus the rate out.

21. Draw a triangle with vertices at $(0,0)$, $(5,0)$ and the remaining side in quadrant I with length 4. Thus the height of the triangle is $h = 4\sin\theta$ and base $b = 5$. Note θ is the angle between the two fixed sides and the final units are in square meters per second. Thus:

$$A = \frac{1}{2}bh = \frac{1}{2}(5)(4\sin\theta) = 10\sin\theta \Rightarrow \frac{dA}{dt} = 10\cos\theta \Rightarrow \frac{dA}{dt} = 10\cos(\frac{\pi}{3}) = \frac{3}{10}$$

22. Draw a circle with radius 100 with center at $(200,0)$. Then the friend is at the origin and the runner is at the point (x,y) on the curve $(x-200)^2 + y^2 = 100^2$. Let s be the distance between the runner and his friend. Thus:

$$s^2 = x^2+y^2 \Rightarrow s^2 = x^2+(100^2-(x-200)^2) \Rightarrow s^2 = 100^2-200^2+400x \Rightarrow 2s\frac{ds}{dt} = 400\frac{dx}{dt}$$

At the time in question, this implies that $\frac{ds}{dt} = \frac{dx}{dt}$. To find $\frac{dx}{dt}$, we need to use some trigometry. Form a triangle between the origin, runner and the center of the circle and let θ be the angle between the x-axis and the edge connecting the runner and the center of the circle. Thus by the law of cosines,

$$200^2 = 200^2 + 100^2 - 2(200)(100)\cos\theta \Rightarrow \cos\theta = \frac{1}{4} \Rightarrow \sin\theta = \frac{\sqrt{15}}{4}$$

By the definition of θ and using opposite interior angles, we obtain $\frac{dx}{dt} = v\sin\theta \Rightarrow \frac{dx}{dt} = \frac{7\sqrt{15}}{4}$ where v is the runner's velocity.

23. Use the fact that $f(x) \approx f(a) + f'(a)(x-a)$.

   (a) Use the function $f(x) = \sqrt{x}$ and $a = 36$. Then $f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(36) = \frac{1}{12}$. Thus $\sqrt{36.1} \approx 6 + \frac{1}{12}(36.1-36) \Rightarrow \sqrt{36.1} \approx \frac{721}{120}$

   (b) Use the function $f(x) = \frac{1}{x}$ and $a = 10$. Then $f'(x) = \frac{-1}{x^2} \Rightarrow f'(1) = \frac{-1}{100}$. Thus $\frac{1}{10.1} \approx \frac{1}{10} - \frac{1}{100}(10.1-10) \Rightarrow \frac{1}{10.1} \approx \frac{99}{1000}$

   (c) Use the function $f(x) = x^6$ and $a = 2$. Then $f'(x) = 6x^5 \Rightarrow f'(2) = 192$. Thus $(1.97)^6 \approx 64 + 192(1.97-2) \Rightarrow (1.97)^6 \approx \frac{1456}{25}$

24. We first find all numbers x such that $f'(x) = 0$ or $f'(x)$ is undefined. These are the **critical numbers** of the function. In the problem, this is what was meant when it ask to find "critical values".

   (a) $f'(x) = 10x - 4$. Thus the critical number is $\left\{\frac{2}{5}\right\}$

   (b) $f'(t) = 6t^2 + 6t - 6$. Thus the critical numbers are $\left\{\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right\}$

(c) $s' = 4t^3+12t^2+4t = 4t(t^2+3t+1)$. Thus the critical numbers are $\left\{0,\ \frac{-3+\sqrt{5}}{2},\ \frac{-3-\sqrt{5}}{2}\right\}$

(d) $f'(r) = \frac{r^2+1-r(2r)}{(r^2+1)} = \frac{1-r^2}{(1+r^2)^2}$. Thus the critical numbers are $\{1\ ,-1\}$ since x^2+1 is never zero.

(e) $g'(x) = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3} = \frac{1}{3}x^{-5/3}(x+2)$. Thus the critical numbers are $\{0,\ -2\}$

(f) $g'(x) = \frac{1}{3}(x^2-x)^{-2/3}(2x-1)$. Thus the critical numbers are $\left\{0,\ 1,\ \frac{1}{2}\right\}$

25. The absolute max and min of a function will always occur at the endpoints of the interval $I = [a, b]$ or critical numbers in I.

(a) $f'(x) = 6x - 12$. The critical number is 2 and so $f(2) = -7$, $f(0) = 5$ and $f(3) = -4$. So $\max_I \{f(x)\} = 5$ and $\min_I \{f(x)\} = -7$.

(b) $f'(x) = 6x^2+6x$. The critical numbers are $\{0,\ 1\}$, $f(0) = 4$, $f(1) = 9$, $f(-1) = 5$ and $f(-2) = 8$. So $\max_I \{f(x)\} = 9$ and $\min_I \{f(x)\} = 4$.

(c) $f'(x) = 2x - \frac{2}{x^2} = 2x^{-2}(x^3-1)$. The critical numbers are $\{0,\ 1\}$ and so $f(1) = 3$, $f\left(\frac{1}{2}\right) = \frac{17}{4}$ and $f(x) = 5$. So $\max_I \{f(x)\} = 5$ and $\min_I \{f(x)\} = 3$.

(d) $f'(x) = \frac{1-x^2}{(x^2+1)^2}$. The critical numbers are $\{-1,\ 1\}$ and so $f(-1) = -\frac{1}{2}$, $f(1) = \frac{1}{2}$, $f(0) = 0$ and $f(2) = \frac{2}{5}$. So $\max_I \{f(x)\} = \dfrac{2}{5}$ and $\min_I \{f(x)\} = -\dfrac{1}{2}$.

(e) $f'(x) = \cos x - \sin x$. The critical number is $\left\{\frac{\pi}{4}\right\}$ and so $f(0) = 1$ and $f\left(\frac{\pi}{4}\right) = \sqrt{2}$. So $\max_I \{f(x)\} = \sqrt{2}$ and $\min_I \{f(x)\} = 1$.

(f) $f'(x) = 1 + 2\sin x$. The critical numbers are $\left\{\frac{-5\pi}{6},\ \frac{-\pi}{6}\right\}$ and so $f\left(\frac{-5\pi}{6}\right) = -\frac{5\pi}{6} + \sqrt{3}$, $f\left(\frac{-\pi}{6}\right) = -\frac{\pi}{6} - \sqrt{3}$, $f(-\pi) = -\pi + 2$ and $f(-\pi) = \pi + 2$. So $\max_I \{f(x)\} = \pi + 2$ and $\min_I \{f(x)\} = -\dfrac{\pi}{6} - \sqrt{3}$.

26. One must check that $f(x)$ is continuous on $[a, b]$ and differentiable (a, b), which is easily done.

(a) $\frac{f(b)-f(a)}{b-a} = \frac{10-6}{1-(-1)} = 2$. $f'(x) = 6x + 2$ so $6c + 2 = 2 \Rightarrow c = 0$

(b) $\frac{f(b)-f(a)}{b-a} = \frac{9-(-1)}{2-0} = 5$. $f'(x) = 3x^2 + 1$ so $3c^2 + 1 = 5 \Rightarrow c = \pm\frac{2}{\sqrt{3}}$. But only $c = \frac{2}{\sqrt{3}}$ is in the interval $[a, b]$.

(c) $\frac{f(b)-f(a)}{b-a} = \frac{1-0}{1-0} = 1$. $f'(x) = \frac{1}{3x^{2/3}}$ so $\frac{1}{3c^{2/3}} = 1 \Rightarrow c = \pm\frac{1}{3^{3/2}}$. But only $c = \frac{1}{3\sqrt{3}}$ is in the interval.

(d) $\frac{f(b)-f(a)}{b-a} = \frac{\frac{2}{3}-\frac{1}{3}}{4-1} = \frac{1}{9}$. $f'(x) = \frac{x+2-x}{(x+2)^2} = \frac{2}{(x+2)^2}$ so

$$\frac{2}{(c+2)^2} = \frac{1}{9} \Rightarrow 18 = (c+2)^2 \Rightarrow c = \pm 3\sqrt{2} - 2$$

But only $c = 3\sqrt{2} - 2$ is in the interval.

27. $f(0) = 3$ and $f(-1) = -8$ and since $f(x)$ is continuous everywhere, the Intermediate Value Theorem gives the existence of a zero. Since $f'(x) = 5x^4 + 10 > 0$ for all x, the zero is unique.

28. Suppose there is such a function. Then $\frac{f(2)-f(0)}{2-0} = \frac{5}{2}$ so by the Mean Value Theorem, $\exists\, c$ such that $f'(c) = \frac{5}{2}$. But this contradicts the fact that $f'(x) \leq 2$ for all x. Thus no such function can exist.

29. A function is increasing when $f'(x) > 0$, decreasing when $f'(x) < 0$, is concave up when $f''(x) > 0$ and concave down when $f''(x) < 0$.

   (a) $f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$. So $f(x)$ is increasing on $(-\infty, -1) \cup (2, \infty)$ and decreasing on $(-1, 2)$. It has a local max at $(-1, 7)$ and a local min at $(2, -32)$.
   $f''(x) = 12x - 6$. So $f(x)$ is concave up on $(-\infty, 1/2)$ and concave down on $(1/2, \infty)$. It has an inflection point at $\left(\frac{1}{2}, -\frac{13}{2}\right)$

   (b) $f'(x) = 4x^3 - 12x = 4x(x^2 - 3)$. So $f(x)$ is increasing on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$ and decreasing on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$. It has a local max at $(0, 0)$ and local mins at $(-\sqrt{3}, -18)$ and $(\sqrt{3}, -18)$.
   $f''(x) = 12x^2 - 12$. So $f(x)$ is concave up on $(-\infty, -1) \cup (1, \infty)$ and concave down on $(-1, 1)$. It has an inflection point at $(-1, -7)$ and $(1, 7)$.

   (c) $h'(x) = 6x(x^2 - 1)^2$. So $f(x)$ is increasing on $(0, 1) \cup (1, \infty)$ and decreasing on $(-\infty, -1) \cup (-1, 0)$ and has no local max and a local min at $(0, -1)$.
   $h''(x) = 6(x^2 - 1)^2 + 24x^2(x^2 - 1) = 6(x^2 - 1)(x^2 + 4x - 1)$. So $f(x)$ is concave up on $(-\infty, -2-\sqrt{5}) \cup (-1, -2+\sqrt{5}) \cup (1, \infty)$ and concave down on $(-2-\sqrt{5}, -1) \cup (-2+\sqrt{5}, 1)$. It has inflection points at $x = \pm 1$ and $x = -2 \pm \sqrt{5}$

   (d) $P'(x) = \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} = (x^2+1)^{-1/2}(2x^2+1)$. So $f(x)$ is increasing everywhere and has no local max or min.
   $P''(x) = \frac{x}{\sqrt{x^2+1}} + \frac{2x\sqrt{x^2+1} - x^2\frac{x}{\sqrt{x^2+1}}}{x^2+1} = \frac{x(2x^2+3)}{(x^2+1)^{3/2}}$. So $P(x)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$. It has an inflection point at $x = 0$.

   (e) $Q'(x) = \sqrt{x+1} + \frac{x}{2\sqrt{x+1}} = (x+1)^{-1/2}(2x+1)$. So $f(x)$ is increasing on $(-1/2, \infty)$ and decreasing on $(-1, -1/2)$. It has a local min at $x = -\frac{1}{2}$.

$Q''(x) = \frac{1}{2\sqrt{x+1}} + \frac{2\sqrt{x+1} - 2x\frac{1}{2}(x+1)^{-1/2}}{4(x+1)} = \frac{3x+4}{4(x+1)^{3/2}}$. So $Q(x)$ is concave up everywhere and no inflection points.

(f) $f'(x) = 1 - x^{-2/3} = x^{-2/3}(x^{2/3} - 1)$. So $f(x)$ is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 1)$. It has a local max at $x = -1$ and local min at $x = 1$. $f''(x) = \frac{2}{3}x^{-5/3}$. So $f(x)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$. It has an inflection point at $x = 0$.

(g) $f'(t) = 1 - \sin t$. So $f(t)$ is increasing everywhere and has no local max or min. $f''(t) = -\cos t$. So $f(t)$ is concave up on $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and concave down on $\left(-2\pi, -\frac{3\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$. $f(t)$ has inflection points at $t = \pm\frac{3\pi}{2}$ and $t = \pm\frac{\pi}{2}$.

30. l'Hôpital's rule is useful for these sort's of limits. The tricks in **b.** , **c.** and **e.** are particually useful.

(a) $\displaystyle\lim_{x\to\infty} \frac{6x^2 + 5x}{(1-x)(2x-3)} = \lim_{x\to\infty} \frac{6x^2 + 5x}{-2x^2 - x - 3} = -3$

(b) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2 + 4x}}{4x + 1} = L \Rightarrow \lim_{x\to\infty} \frac{x^2 + 4x}{(4x+1)^2} = \lim_{x\to\infty} \frac{x^2 + 4x}{16x^2 + 8x + 1} = L^2 \Rightarrow L^2 = \frac{1}{16}$. Thus the limit is $\frac{1}{4}$.

(c) $\displaystyle\lim_{x\to\infty} \sqrt{x^2 + 3x + 1} - x = \lim_{x\to\infty} \frac{x^2 + 3x + 1 - x^2}{\sqrt{x^2 + 3x + 1} + x}$. By using l'Hôpital's Rule, the limit is same as $\displaystyle\lim_{x\to\infty} \frac{3}{\frac{2x+3}{2\sqrt{x^2+3x+1}} + 1}$.
Now $\displaystyle\lim_{x\to\infty} \frac{2x+3}{2\sqrt{x^2 + 3x + 1}} = L \Rightarrow \lim_{x\to\infty} \frac{4x^2 + 12x + 9}{4(x^2 + 3x + 1)} = L^2$ Thus $\Rightarrow L = 1$. Hence the limit is $\frac{3}{2}$.

(d) $\displaystyle\lim_{x\to\infty} \frac{1 - \sqrt{x}}{1 + \sqrt{x}} = \lim_{x\to\infty} \frac{\frac{-1}{2\sqrt{x}}}{\frac{1}{2\sqrt{x}}} = -1$ by l'Hôpital's Rule.

(e) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = L \Rightarrow \lim_{x\to\infty} \frac{4x^2 + 1}{x^2 + 2x + 1} = L^2 \Rightarrow L = 2$

31. $xy = 100$ and $S = x + y$. Subsituting for y yields $S = x + \frac{100}{x} \Rightarrow S' = 1 - \frac{100}{x^2}$. The critical number with $x = 10$ is a minimum and so $x = y = 10$.

32. $A = xy = \frac{3}{2}$. The perimeter of the fence P is given by $P = 2y + 3x \Rightarrow P = \frac{3}{x} + 3x$. Minimizing this function yields $P' = -\frac{3}{x^2} + 3$ and the critical number $x = 1$ yields a minimum. Hence $x = 1$ and $y = \frac{3}{2}$. The units are in feet.

33. $V = 2x^2y$ and the cost function is given by $C = 10(2x^2) + 6(2)(2xy) + 6(2)(xy) = 20x^2 + 36xy$ since there is one side of area $2x^2$ at \$10 per square foot, 2 sides of area $2xy$ and 2 sides of area xy at \$6 per quare foot. Thus $C = 20x^2 + \frac{180}{x} \Rightarrow C' = 40x - \frac{180}{x^2}$. The critical number $x = \frac{6^{2/3}}{2}$ corresponds to the minimum which makes the minimal cost, C_{min} ≈\$163.54

34. Don't forget $+C$!

 (a) $F(x) = 2x^3 - 4x^2 + 3x + C$

 (b) $F(x) = x - \frac{1}{4}x^4 + \frac{5}{6}x^6 - \frac{3}{8}x^8 + C$

 (c) $F(x) = 4x^{5/4} - 4x^{7/4} + C$

 (d) $F(x) = -\frac{5}{2}x^{-8} + C$

 (e) $F(x) = -\frac{3}{x} + \frac{5}{x^3} + C$

35. For the solutions, C α, β and γ are arbitrary constants.

 (a) $\Rightarrow f'(x) = 3x^2 + 4x^3 + \alpha \Rightarrow f(x) = x^3 + x^4 + \alpha x + \beta$

 (b) $\Rightarrow f'(x) = x + \frac{5}{9}x^{9/5} + \alpha \Rightarrow f(x) = \frac{1}{2}x^2 + \frac{25}{126}x^{14/5} + \alpha x^2 + \beta$

 (c) $\Rightarrow f''(t) = 20t^3 + \alpha \Rightarrow f'(t) = 5t^4 + \alpha t + \beta \Rightarrow f(t) = t^5 + \alpha t^2 + \beta t + \gamma$

 (d) $\Rightarrow f''(t) = \frac{1}{2}t^2 - \frac{2}{3}t^{3/2} + \alpha \Rightarrow f'(t) = \frac{1}{6}t^3 - \frac{4}{15}t^{5/2} + \alpha t + \beta$
 $\Rightarrow f(t) = \frac{1}{24}t^4 - \frac{8}{105}t^{7/2} + \alpha t^2 + \beta t + \gamma$

 (e) $\Rightarrow f(x) = x - \frac{1}{x} + C$. Since $f(1) = 2$, we have that $c = 2$ hence $f(x) = x - \frac{1}{x} + 2$.

 (f) $\Rightarrow f(x) = 3\sin x - 5\cos x + C$. Since $f(0) = 4$, we have that $c = 9$ and so $f(x) = 3\sin x - 5\cos x + 9$

 (g) $\Rightarrow f'(x) = 4x^3 - 3x^2 + \frac{1}{2}x^2 + \alpha \Rightarrow f(x) = x^4 - x^3 = \frac{1}{6}x^3 + \alpha x + \beta$. Since $f(0) = 1$, we have that $\beta = 1$ and since $f(2) = 11$ we have $\alpha = \frac{1}{3}$. Thus $f(x) = x^4 - \frac{5}{6}x^3 + \frac{1}{3}x + 1$.

36. Use the formula that the Riemann Sum is the sum of the area of the rectangles. In each case, assume you use n, an aribtrary number of rectangles. Note that I have used Right Handed Sums.

 (a) We have that the width of each rectangle is $\frac{8}{n}$ and the height is given by the function value. Thus the area is given by $\lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{8}{n}\sqrt[3]{\frac{k}{n}}\right)$.

 (b) The width of each rectangle is $\frac{\pi}{n}$ and so the area is given by

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{\pi}{n}\left[\left(\pi + \frac{k\pi}{n}\right) + \sin\left(\pi + \frac{k\pi}{n}\right)\right]\right)$$

37. I will use a right hand sum to estimate the areas. Note that for $n = \infty$, this is given by the definite integral.

   (a) i. The width of each rectangle is 2 and the heights are 4 and 10 so the area is 28.

      ii. The width of each rectangle is 1 and the heights are 1, 4, 7 and 10 so the area is 22.

      iii. $\displaystyle\int_{-1}^{3} (1+3x)\,dx = x + \frac{3}{2}x^2 \Big|_{-1}^{3} = 3 + \frac{3}{2}(9) - \left(-1 + \frac{3}{2}\right) = 16$

   (b) i. The width of each rectangle is 1 and the heights are 1 and -2 so the area is -1.

      ii. The width of each rectangle is $\frac{1}{2}$ and the heights are $\frac{3}{4}$, 1 $\frac{-1}{4}$ and -2 so the area is $\frac{-1}{4}$.

      iii. $\displaystyle\int_{0}^{2} (2-x^2)\,dx = 2x - \frac{1}{3}x^3 \Big|_{0}^{2} = 4 - \frac{8}{3} = \frac{4}{3}$

   (c) i. The width of each rectangle is 2 and the heights are 17 and 129 so the area is 292.

      ii. The width of each rectangle is 1 and the heights are 3, 17, 55 and 129 so the area is 204.

      iii. $\displaystyle\int_{0}^{4} (1+2x^3)\,dx = x + \frac{1}{2}x^4 \Big|_{0}^{4} = 4 + 128 = 132$

38. Drawing a picture will help.

   (a) The area of the square is 9 and the area of the triangle is 4 so the integral is 13.

   (b) The area of the rectangle is 3 and the area of the semicircle is $\frac{9\pi}{4}$ so the integral is $3 + \frac{9\pi}{4}$.

   (c) The area of the "outside" rectangles are both $-\frac{1}{2}$ and the area of the "inside" rectangle is 1 so the integral is 0.

39. This is the Fundamental Theorem of Calculus.

   (a) $g'(x) = \sqrt{1+2x}$

   (b) $g'(y) = y^2 \sin y$

   (c) $f(x) = -\cos(x^2)$ since $-F(x) = \displaystyle\int_{2}^{x} \cos(t^2)\,dt$

   (d) $-\tan x$

   (e) $2x\sqrt{1+x^4}$

(f) $3x^2\sin(x^3)$

40. (a) $\Rightarrow= \frac{5}{3}x^3 - 2x^2 + 3x\Big|_1^2 = \frac{26}{3}$

(b) $\Rightarrow= \frac{1}{10}x^10 - \frac{1}{3}y^6 + \frac{3}{2}y^2\Big|_0^1 = \frac{19}{15}$

(c) $\Rightarrow= \frac{1}{3}t^3 + t^{-1}\Big|_1^2 = \frac{11}{6}$

(d) $\Rightarrow= \displaystyle\int_0^2 (x^6 - 2x^3 + 1)\;dx = \frac{1}{7}x^7 - \frac{1}{2}x^4 + x\Big|_0^2 = \frac{86}{7}$

(e) $\Rightarrow= \displaystyle\int_{-1}^1 -3x^2 + x + 2\;dx = -x^3 + \frac{1}{2}x^2 + 2x\Big|_{-1}^1 = 2$

(f) $\Rightarrow= -\csc x\Big|_{\pi/3}^{\pi/2} = \frac{2\sqrt{3}}{3} - 1$

41. The total area bound by a curve $y = f(x)$ is given by $\displaystyle\int_a^b |f(x)|\;dx$

(a) $\displaystyle\int_{-2}^{-4} \big|3x - 1\big|\;dx = \int_{-2}^{1/3} 1 - 3x\;dx + \int_{1/3}^{4} 3x - 1\;dx$

$\Rightarrow= x - \frac{3}{2}x^2\Big|_{-2}^{1/3} + \frac{3}{2}x^2 - x\Big|_{1/3}^{4} = \frac{85}{3}$

(b) $\displaystyle\int_{-3}^{3} |x^2 - x - 2|\;dx = \int_{-3}^{-1} x^2 - x - 2\;dx + \int_{-1}^{2} 2 - x - x^2\;dx + \int_{2}^{3} x^2 - x - 2\;dx$

$\Rightarrow= \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\Big|_{-3}^{-1} + 2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\Big|_{-1}^{2} + \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\Big|_{2}^{3} = 12$

42. These are done using a u substitution. Don't forget the $+C$!

(a) Let $u = 1 - x^4 \Rightarrow = -\frac{1}{4}\int u^5\, du = -\dfrac{1}{24}u^6 + C = -\dfrac{1}{24}(1-x^4)^6 + C$

(b) Let $u = 2 - x \Rightarrow = -\int u^6\, du = -\dfrac{1}{7}u^7 = -\dfrac{1}{7}(2-x)^7 + C$

(c) Let $u = x^2 + 1 \Rightarrow = \frac{1}{2}\int u^{3/2}\, du = \dfrac{1}{5}u^{5/2} = \dfrac{1}{5}(x^2+1)^{5/2} + C$

(d) Let $u = 1 - 3x \Rightarrow = -\frac{1}{3}\int \dfrac{1}{u^4}\, du = \dfrac{1}{9}u^{-3} + C = \dfrac{1}{9}(1-3x)^{-3} + C$

(e) Let $u = 3 - 5x \Rightarrow = -\frac{1}{5}\int u^{1/5}\, du = -\dfrac{1}{6}u^{6/5} + C = -\dfrac{1}{6}(3-5x)^{6/5} + C$

(f) Let $u = 3x \Rightarrow = \frac{1}{3}\int \sec^2 u\, du = \dfrac{1}{3}\tan u + C = \dfrac{1}{3}\tan(3x) + C$

(g) Let $u = 1 + \sqrt{x} \Rightarrow = 2\int u^9\, du = \dfrac{1}{5}u^{10} + C = \dfrac{1}{5}(1+\sqrt{x})^{10} + C$

(h) Let $u = \frac{\pi}{x} \Rightarrow = -\frac{1}{\pi}\int \cos u\, du = -\dfrac{1}{\pi}\sin u + C = -\dfrac{1}{\pi}\sin\left(\dfrac{\pi}{x}\right) + C$

(i) Let $u = \sec x \Rightarrow = \int u^2\, du = \dfrac{1}{3}u^3 + C = \dfrac{1}{3}\sec^3 x + C$

43. These are also done using a u substitution. Don't forget to change the limits of integration!

(a) Let $u = 4 + 3x \Rightarrow = \frac{1}{3}\int_4^{25} \sqrt{u}\, du = \dfrac{2}{9}u^{3/2}\Big|_4^{25} = \dfrac{112}{9}$

(b) Let $u = x^2 \Rightarrow = \frac{1}{2}\int_0^{\pi} \cos u\, du = \dfrac{1}{2}\sin u\Big|_0^{\pi} = 0$

(c) Let $u = 4t \Rightarrow = \frac{1}{4}\int_0^{\pi} \sin u\, du = -\dfrac{1}{4}\cos u\Big|_0^{\pi} = \dfrac{1}{2}$

(d) Let $u = 1 + \frac{1}{x} \Rightarrow = -\int_2^{5/4} \sqrt{u}\, du = -\dfrac{2}{3}u^{3/2}\Big|_2^{5/4} = -\dfrac{2}{3}\left(\dfrac{5\sqrt{5}}{8} - 2\sqrt{2}\right)$

(e) Let $u = \pi t \Rightarrow = \frac{1}{\pi}\int_0^{\pi} \cos u\, du = -\dfrac{1}{\pi}\sin u\Big|_0^{\pi} = 0$

(f) Let $u = a^2 - u^2 \Rightarrow = -\frac{1}{2}\int_{a^2}^{0} \sqrt{u}\, du = \dfrac{1}{2}\int_0^{a^2} \sqrt{u}\, du = \dfrac{1}{3}u^{3/2}\Big|_0^{a^2} = \dfrac{a^3}{3}$

44. Observe that $f(x) = \frac{x^2 \sin x}{1+x^6}$ is an odd function since $f(-x) = -f(x)$ so $\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6}\, dx =$ 0

45. Let $u = x^2$. Then $\int_0^1 x\sqrt{1-x^4}\, dx = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\, du$. The last integral is the area of a quater-circle with radius 1. Thus the integral is $\frac{\pi}{8}$.

46. Let $u = x^2$ then $\int_0^3 xf(x)\, dx = \frac{1}{2}\int_0^9 f(u)\, du = 2$

47. Observe that $\int_2^{2+h} \sqrt{1+t^3}\, dt = \int_0^{2+h} \sqrt{1+t^3}\, dt - \int_0^2 \sqrt{1+t^3}\, dt = f(2+h) - f(2)$ where $f(x) = \int_0^x \sqrt{1+t^3}\, dt$. Thus by the definition of the derivative:

$$\lim_{h\to 0} \frac{1}{h}\int_2^{2+h} \sqrt{1+t^3}\, dt = f'(2)$$

By the Fundamental Theorem of Calculus, $f'(x) = \sqrt{1+x^3}$ and so

$$\lim_{h\to 0} \frac{1}{h}\int_2^{2+h} \sqrt{1+t^3}\, dt = \sqrt{1+8} = 3$$