RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Topology-Geometry

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

January 2026

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1. Suppose that $p : X \to Y$ is a quotient map between topological spaces with

(1) Y connected and

(2) $p^{-1}(y)$ connected for each $y \in Y$.

Prove that X is connected.

Problem 2.

(a) Prove that the fundamental group of the wedge of two circles (that is, two circles joined at a single point) is the free group on two generators.

(b) Let X be the union of the unit circle and the x-axis inside of $\mathbb{R}^2$. That is,

$$X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2 \mid y = 0\}.$$

Determine the fundamental group of X.

Problem 3. Suppose that X is a path connected, locally path connected space with $\pi_1(X)$ a finite group and S^1 is the circle. Prove that any continuous map $f : X \to S^1$ is nullhomotopic.

Problem 4. Define $F : \mathbb{R}^6 \to \mathbb{R}^2$ by

$$F(x, y, z, u, v, w) = (x^2 + y^2 + z^2 - 1,\ xu + yv + zw).$$

Show that $F^{-1}(0, 0)$ is an embedded submanifold of $\mathbb{R}^6$.

Problem 5. On $\mathbb{R}^3$ with the standard coordinates (x, y, z) consider vector fields

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}.$$

(a) Compute $[X, Y]$.

(b) Show that there is no surface S in $\mathbb{R}^3$ such that both X and Y are tangent to S.

Problem 6. Let k, l be positive integers, and let M be a manifold (without boundary) of dimension $k+l$. Assume that $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$ are two differential forms, and that α is **closed** and β is **exact**.

(a) Assume, in addition, that M is compact. Show that there exists a point in M where $\alpha \wedge \beta$ vanishes.

(b) Show that the conclusion of the previous part is not necessarily true if M is not compact by giving an example of a noncompact M, closed α, and exact β such that $\alpha \wedge \beta$ is nowhere 0.