RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

# Topology-Geometry

## Ph.D. Preliminary Examination

## Department of Mathematics

## University of Colorado

January, 2025

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

| Problem | 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|
| Points | 17 | 17 | 17 | 17 | 17 | 17 | 102 |
| Score | | | | | | | |

**Problem 1.** Suppose that $p : X \to Y$ is a map between topological spaces that satisfies

(a) $p(F)$ is closed in $Y$ whenever $F$ is closed in $X$ and

(b) $p^{-1}(y)$ is compact for each $y \in Y$.

Prove that if $Y$ is compact, then $X$ is also compact.
(Hint: Start by proving that if $U$ is an open set in $X$ containing $p^{-1}(y)$, then there is an open set $W$ in $Y$ containing $y$ such that $p^{-1}(W) \subseteq U$).

**Problem 2.** Let $T = S^1 \times S^1$ be the torus. The topology on $T$ is the standard one, which is given by the product topology (where we take the standard topology on each copy of $S^1$). Let $(b_0, b_0) \in T$ (where $b_0 \in S^1$) be the base point. Consider the following,

(a) Prove that if $p_1 : X_1 \to Y_1$ and $p_2 : X_2 \to X_2$ are covering maps, then $p_1 \times p_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a covering.

(b) Show that $\pi_1(T, (b_0, b_0)) \cong \mathbb{Z} \times \mathbb{Z}$.

(c) Find a covering space of $T$ corresponding to the trivial subgroup of $\pi_1(T, (b_0, b_0)) \cong \mathbb{Z} \times \mathbb{Z}$.

(d) Find a covering space of $T$ corresponding to the subgroup of $\pi_1(T, (b_0, b_0)) \cong \mathbb{Z} \times \mathbb{Z}$ generated by $(m, 0)$ where $m$ is a positive integer.

(e) Find a covering space of $T$ corresponding to the subgroup of $\pi_1(T, (b_0, b_0)) \cong \mathbb{Z} \times \mathbb{Z}$ generated by $(m, 0)$ and $(0, n)$ where $m$ and $n$ are positive integers.

Note: Any time you are asked to find a covering space, you must find the space $X$ and the covering map $p : X \to T$. Use well-known covering maps of $S^1$ and the first part of the question to justify that the maps you have picked are covering maps.

**Problem 3.** Recall the following definition: Suppose that $X$ is a topology space and $A$ a topological subspace of $X$. We say that $r : X \to A$ is a retraction of $X$ onto $A$ if $r$ is a continuous map such that $r|_A$ is the identity map of $A$. In this case, we call $A$ a retract of $X$.

(a) Prove that if $A$ is a retract of $X$ and $j : A \to X$ denotes the inclusion of $A$ into $X$, then $j_* : \pi_1(A, a) \to \pi_1(X, j(a))$ is injective (where $a \in A$).

(b) Let $D^2$ be the two dimensional disk (with its standard topology). The boundary of $D^2$ is $S^1$ (where $S^1$ has its standard topology). Does there exist a retraction from $S^1 \times D^2$ to $S^1 \times S^1$? (Justify your answer with the construction of a retraction or a proof that one cannot exist).

(c) Does there exist a retraction from $X = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}$ to $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$? (Justify your answer with the construction of a retraction or a proof that one cannot exist).

**Problem 4.** On $\mathbb{R}^3$ with the standard coordinates $(x, y, z)$ consider a differential form $\omega = 2zx\, dy \wedge dz + (1 - z^2) dx \wedge dy$.

(a) Determine if $\omega$ is closed. Is it exact?

(b) Let $S$ be a surface in $\mathbb{R}^3$ given by the equation $x^2 + y^2 + z^4 = 1$ oriented by the outward normal, and let $N \subset S$ be the set of points in $S$ where $z \geq 0$. Evaluate $\int_N \omega$.

**Problem 5.** On $\mathbb{R}^3$ with the standard coordinates $(x, y, z)$ consider vector fields

$$X = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}$$

(a) Compute $[X, Y]$.

(b) Show that there is no surface $S$ in $\mathbb{R}^3$ such that both $X$ and $Y$ are tangent to $S$.

**Problem 6.** The set of $M_{2\times 2}(\mathbb{R})$ of $2 \times 2$ real matrices is naturally a vector space and hence a smooth manifold.

(a) Prove that the set $SL_2(\mathbb{R})$ of matrices with determinant 1 is an embedded submanifold of $M_{2\times 2}(\mathbb{R})$.

(b) By the previous part, the tangent space $T_e SL_2(\mathbb{R})$ at the identity matrix $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ can be naturally regarded as a subspace of $T_e M_{2\times 2}(\mathbb{R}) = M_{2\times 2}(\mathbb{R})$. Give an explicit description of this subspace.

(c) Compute differential at $e$ of the map $F\colon SL_2(\mathbb{R}) \to SL_2(\mathbb{R})$ given by

$$F(A) := A \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} A^{-1}$$