RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Topology-Geometry

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

January, 2024

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1.

(a) Let X and Y be topological spaces. Let $f\colon X \to Y$ be a continuous function. Suppose that X is compact and Y is Hausdorff. Prove that f is a topological embedding if and only if f is injective. (Recall that a topological embedding is defined to be a continuous function $f\colon X \to Y$ such that the function $g\colon X \to f(X)$, $g(x) = f(x)$, obtained by restricting the range of f is a homeomorphism, where $f(X)$ is given the subspace topology in Y.)

(b) Let $\mathbb{R}$ have the standard topology and $[0, 2\pi) = \{x \in \mathbb{R} : 0 \leq x < 2\pi\}$ have the subspace topology. Prove that the function

$$h\colon [0, 2\pi) \to \mathbb{R}^2$$
$$\theta \mapsto h(\theta) = (\cos(\theta), \sin(\theta))$$

is a continuous injection which is not a topological embedding. You may use standard facts about trigonometric functions without proof, including the fact that $\sin, \cos : \mathbb{R} \to \mathbb{R}$ are continuous functions.

(c) Let $\mathbb{R}^2$ have the standard topology and

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

be the unit circle equipped with the subspace topology. Is the space $[0, 2\pi)$ as in (b) homeomorphic to $\mathbb{S}^1$? Prove your claim.

Problem 2. Let E and B be topological spaces.

(a) Let $p\colon E \to B$ be a covering map and $b \in B$. Prove that the subspace $p^{-1}(\{b\}) \subset E$ has the discrete topology.

(b) Let $\mathbb{S}^2$ be the unit sphere topologized as a subspace of $\mathbb{R}^3$. Suppose that $p\colon E \to \mathbb{S}^2$ is an n-fold covering map for some $n \geq 1$. Identify the space E up to homeomorphism. (Your answer should be specific. It should depend on n, but not on the particular n-fold covering p.)

Problem 3.

(a) Let K be the Klein bottle and W be a genus 2 surface. Give a presentation for the fundamental group of the connected sum $K \# W$. (The gluing diagrams for the spaces K and W are in Figure 1. Recall that the connected sum is obtained by removing an open disk from each surface and gluing the two resulting boundaries together.)

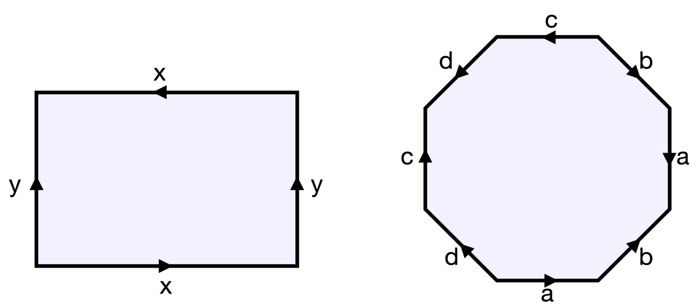

FIGURE 1. Gluing diagrams for the Klein bottle K (left) and the genus 2 surface W (right).

(b) Let $T \cong \mathbb{S}^1 \times \mathbb{S}^1$ be the torus. Prove that the spaces $T \# W$ are $K \# W$ are not homotopy equivalent.

Problem 4.

(a) Let θ and ϱ be smooth 3-foms on $\mathbb{S}^7$. Prove that

$$\int_{\mathbb{S}^7} \theta \wedge d\varrho = \int_{\mathbb{S}^7} d\theta \wedge \varrho.$$

(b) Let M be a smooth manifold and ω a differential form on M. Prove that if ω has even degree then $\omega \wedge d\omega$ is exact.

Problem 5. Consider the map $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x, y, z) = (z^2 - xy, x^2 + y^2)$.

(a) Find all the critical points of f.

(b) Show that for $b > 2a > 0$ the pre-image $f^{-1}(a, b)$ is a non-empty smooth one-dimensional sub-manifold of $\mathbb{R}^3$.

Problem 6. Determine (with justification) whether or not each of the following smooth maps is an immersion, a submersion, and/or an embedding.

(a) $f : \mathbb{S}^1 \to \mathbb{R}$ given by $f(x, y) = y$, where $\mathbb{S}^1 \subset \mathbb{R}^2$ is the unit circle.

(b) $g : \mathbb{S}^2 \to \mathbb{RP}^2$ given by $g(x) = [x]$, where $\mathbb{RP}^2$ is the real projective space of dimension 2 defined as the quotient space $\mathbb{S}^2/\sim$ under the equivalence relation $x \sim -x$, and $[x] \in \mathbb{RP}^2$ is the equivalence class of $x \in \mathbb{S}^2$.

(c) $h : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ given by $h([t]) = (\cos 2\pi t, \sin 2\pi t, 0)$, where $\mathbb{R}/\mathbb{Z}$ is the quotient of $\mathbb{R}$ by the equivalence relation $x \sim x + n$ for all $n \in \mathbb{Z}$, with manifold structure defined by the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$.

Hints: The notion of an embedding is meant here in the differentiable sense. You do not need to show smoothness of the maps f, g, h.