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Topology Prelim — January 2023

All problems carry the same weight. Justify your claims.

ADDITIONAL INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.
2. Label each answer sheet with the problem number.
3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Define an equivalence relation on $\mathbb{R}^n$ by $x \sim y$ if and only if $f(x) = f(y)$. Let X be the quotient space with the quotient toipology.

   (a) Show that X is always Hausdorff.

   (b) Must X be path-connected? Prove by using known theorems or provide a counterexample.

   (c) Must X be compact? Prove by using known theorems or provide a counterexample.

   (d) Is (a) true if f is not continuous? Prove or provide a counterexample.

2. Let $B^2 \subseteq \mathbb{R}^2$ be the set of vectors of length less than or equal to one, and $S^1 \subseteq B^2$ be those vectors of length exactly one. In this problem, you may assume without proof that the map $\pi : S^1 \times [0,1] \to B^2$ given by $\pi(x,t) = (1-t)x$ is a quotient map.

   (a) Prove that if $h : S^1 \to S^1$ is a continuous map which is nulhomotopic, then there exists a continuous map $f : B^2 \to S^1$ such that $f(x) = h(x)$ for all $x \in S^1$.

   (b) Prove that a continuous nulhomotopic map $h : S^1 \to S^1$ has a fixed point.

3. Let V be the real vector space of infinite tuples $(x_0, x_1, x_2, \ldots)$ such that all but finitely many of the x_i are 0. Let

$$S^\infty = \{(x_0, x_1, x_2, \ldots) \in V \mid \sum x_i^2 = 1\}$$

   Include the n-dimensional sphere S^n in S^∞ as the subspace where $x_{n+1} = x_{n+2} = \cdots = 0$. Call a subset U of S^∞ open if $U \cap S^n$ is open in S^n for all integers $n \geq 0$.

   (a) Show that the complement of the point $p = (1,0,0,\ldots)$ in S^∞ is contractible.

   (b) Let $f : S^\infty \to S^\infty$ be the function $f(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots)$. Show that f is homotopic to the identity map.

   (c) Conclude that S^∞ is contractible.

   (d) For every positive integer m, construct a topological space with contractible universal cover and fundamental group $\mathbb{Z}/m\mathbb{Z}$. (Hint: make use of the unit sphere in the complex version of V.)

4. Suppose that G is a connected Lie group with identity g. Let H be its universal cover and let h be a point of H such that $\pi(h) = g$. Show that there is a unique Lie group structure on H such that the identity element is h and the projection $\pi : H \to G$ is a homomorphism of Lie groups. Remember to describe the manifold structure of H.

5. Let S be the sphere in $\mathbb{R}^{n+1}$ defined by the equations $x_0^2 + \cdots + x_n^2 = 1$ and let ω be the following n-form on $\mathbb{R}^{n+1}$:

$$\omega = \sum_{k=0}^{n} (-1)^k x_k dx_0 \wedge \cdots \widehat{dx_k} \cdots \wedge dx_n$$

   The hat means the factor is omitted.

   (a) Compute $d\omega$.

(b) Show that the restriction $\omega\big|_S$ of ω to S is closed.

(c) Show that $\int_S \omega \neq 0$.

(d) Conclude that the restriction of ω to S is not exact.

6. Let $M \simeq \mathbb{R}^{n^2}$ be the space of $n \times n$ matrices with real entries. Let $S \simeq \mathbb{R}^{n(n+1)/2}$ be the space of symmetric $n \times n$ matrices. (Recall that a matrix A is symmetric if $A^t = A$, where $(-)^t$ denotes the transpose.)

   (a) Show that the function $F : M \to S$ given by $F(A) = AA^t$ (where A^t is the transpose of A) is a submersion near the identity matrix, I.

   (b) Deduce (or prove directly) that F is submersive near all points of $F^{-1}(I)$.

   (c) Conclude that the group O_n of $n \times n$ orthogonal matrices is a submanifold of M. (Recall that a matrix A is orthogonal if $A^t = A^{-1}$.)

   (d) Characterize the tangent space to O_n at the origin as a subspace of the tangent space of M.