RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

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INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Q.1 Suppose that $\{X_i\}_{i \in I}$ is a collection of topological spaces.

(a) Prove that for each $j \in I$, the projection map $p_j : \prod_{i \in I} X_i \to X_j$ defined via $(x_i)_{i \in I} \mapsto x_j$ is continuous with respect to both the product and the box topology. (Recall that the box topology on $\prod_{i \in I} X_i$ is the topology generated by the basis

$$\beta_{box} = \left\{ \prod_{i \in I} U_i \mid U_i \text{ is open in } X_i \text{ for each } i \in I \right\}.)$$

(b) Suppose that Y is a topological space and $f : Y \to \prod_{i \in I} X_i$ is a function. Prove that f is continuous with respect to the product topology if and only if for each $j \in I$, $p_j \circ f : Y \to X_j$ is continuous.

(c) Suppose that $I = \mathbb{N}$ and $X_i = \mathbb{R}$ for each $i \in \mathbb{N}$. Prove that the function $f : \mathbb{R} \to \prod_{i \in I} X_i$ defined via $t \mapsto (t, t, t, \ldots)$ is continuous with respect to the product topology.

(d) Prove that the function $f : \mathbb{R} \to \prod_{i \in I} X_i$ defined in the previous part is **not** continuous with respect to the box topology.

Q.2 Recall that S^n denotes the n-sphere and B^n denotes the closed n-ball. Suppose that to every continuous map $h : S^n \to S^n$ we have assigned an integer called its degree (denoted by $\deg(h)$) and that the degree has the following properties:

(i) Homotopic maps have the same degree.

(ii) $\deg(h \circ k) = \deg(h) \cdot \deg(k)$

(iii) The identity map has degree one, any constant map has degree zero and the reflection map $h(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n, -x_{n+1})$ has degree minus one.

Note: You do not need to prove the degree exists, rather you can assume it exists and has the three properties (i)-(iii).

Prove the following:

(a) There is no retraction $f : B^{n+1} \to S^n$.

(b) If $h : S^n \to S^n$ has degree different than $(-1)^{n+1}$ then h has a fixed point.

(c) If $h : S^n \to S^n$ has degree different than one, then there exists $x_0 \in S^n$ such that $h(x_0) = -x_0$.

Q.3 Recall that a topological group, G, is a topological space that is also a group with the property that the maps $G \times G \to G$ defined by multiplication and $G \to G$ defined via $g \mapsto g^{-1}$ are continuous functions.

Prove the following theorem:

Suppose that $\tilde{G}$ and G are connected topological groups and the map $\rho : \tilde{G} \to G$ is both a covering map and a group homomorphism. Then $\tilde{G}$ is abelian if and only if G is abelian.

Q.4 Let $M = \mathbb{R}^2$, and consider the vector fields

$$X = (x+1)\,\frac{\partial}{\partial x} - (y+1)\,\frac{\partial}{\partial y}, \qquad Y = (x+1)\,\frac{\partial}{\partial x} + (y+1)\,\frac{\partial}{\partial y}$$

on M.

(a) Show that there exist local coordinates (s,t) in some neighborhood U of the point $(1,0)$ such that the restrictions of X and Y to U are given by $X = \frac{\partial}{\partial s}$ and $Y = \frac{\partial}{\partial t}$.

(b) Find such coordinates explicitly, and verify directly that they satisfy the conditions $X = \frac{\partial}{\partial s}$ and $Y = \frac{\partial}{\partial t}$.

Q.5 Let $a, b \in \mathbb{R}$, and consider the subset S of $\mathbb{R}^3$ defined by the equations

$$x^2 - z^2 = a^2, \qquad x^2 + y^2 + z^2 = b^2.$$

(a) Show that if $a, b \neq 0$ and $a^2 \neq b^2$, then S is a regular submanifold of $\mathbb{R}^3$.

(b) Describe the set S when $a = b = 1$. Is it a regular submanifold of $\mathbb{R}^3$?

Q.6 Let M be a compact, oriented n-dimensional manifold without boundary. A *volume form* on M is a nowhere vanishing n-form Ω on M with the property that $\displaystyle\int_M \Omega > 0$.

(a) Show that every volume form Ω on M is closed.

(b) Show that a volume form Ω on M cannot be exact.