RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Topology-Geometry

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

August 2025

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1. Assume that $\mathbb{R}^n$, with $n \geq 1$, and $\mathbb{R}$ have the standard topology, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Define an equivalence relation on $\mathbb{R}^n$ by $x \sim y$ if and only if $f(x) = f(y)$. Endow the set X of the equivalence classes of this equivalence relation with the quotient topology.

(a) Show that X is Hausdorff. (You might find it helpful to show that the quotient map is open.)

(b) Must X be path-connected? Either prove this claim by using known theorems or provide a counterexample.

(c) Must X be compact? Either prove this claim by using known theorems or provide a counterexample.

Problem 2.

(a) Assume that $\mathbb{R}$ has the standard topology.

i. Prove that every countable subset A of $\mathbb{R}$ with more than one element is disconnected.

ii. Prove that $\mathbb{R} \setminus A$ is also disconnected

iii. Prove that the intervals are connected subsets of $\mathbb{R}$.

(b) Assume that $\mathbb{R}$ has the topology generated by the family of intervals $\{[a, b) | a, b \in \mathbb{R}\}$. Is every countable subset A of $\mathbb{R}$ with more than one element disconnected?

Problem 3. Let $\mathbb{R}^3$ be endowed with the standard topology.

(a) Compute the fundamental group of the following subspace S of $\mathbb{R}^3$:

$$S := \left\{x^2 + y^2 + (z-1)^2 = 1\right\} \bigcup \left\{y^2 + (z+1)^2 = 1, x = 0\right\} \bigcup \left\{y^2 + (z+1/2)^2 = 1/4, x = 0\right\}.$$

(b) Compute the universal cover (i.e., a connected cover with trivial fundamental group) of the following subspace W of $\mathbb{R}^3$:

$$W := \left\{x^2 + y^2 + (z-1)^2 = 1\right\} \bigcup \left\{y^2 + (z+1)^2 = 1, x = 0\right\}.$$

(c) Can the real projective space $\mathbb{R}P^n$, $n \geq 2$ have a connected 3-fold cover? (Note: $\mathbb{R}P^n$ is the quotient space of the standard sphere S^n by the antipodal map.) Explain.

Problem 4. Let M be a manifold and let $\omega^1, \ldots, \omega^k$ be smooth 1-forms on M such that $\omega^1\big|_p, \ldots, \omega^k\big|_p$ are linearly independent for each $p \in M$. Suppose that $\alpha^1, \ldots, \alpha^k$ are smooth 1-forms on M such that $\sum \alpha^i \wedge \omega^i = 0$. Show that each α^i is a linear combination of $\omega^1, \ldots, \omega^k$ with *smooth* coefficients.

Problem 5. Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the 2-sphere, with orientation corresponding to the outward pointing normal vector. Let ω be the following differential form:

$$\omega = \begin{cases} \frac{dy \wedge dz}{x} & x \neq 0 \\ \frac{dz \wedge dx}{y} & y \neq 0 \\ \frac{dx \wedge dy}{z} & z \neq 0 \end{cases}$$

(a) Show that ω is a well-defined 2-form on M.

(b) Find $\int_M \omega$.

(c) Determine whether or not ω is exact, with justification.

Problem 6. Classify 1-dimensional manifolds following the steps below:

(a) Show that every orientable 1-manifold is diffeomorphic to either the real line or to the circle.

(b) Show that every 1-dimensional manifold is orientable. (Hints: first show that simply connected 1-dimensional manifolds are orientable, then show that any orientation reversing diffeomorphism of $\mathbb{R}$ must fix a point.)