RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Topology-Geometry

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

August, 2024

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1. Let $n \geq 1$ be an integer. Recall that a $n \times n$ real matrix is orthogonal if multiplication by the matrix preserves length of vectors in $\mathbb{R}^n$. Let $O(n)$ be the group of $n \times n$ orthogonal matrices. Give $O(n)$ the topology from the space of $n \times n$ real matrices (which is homemorphic to $\mathbb{R}^{n^2}$). Show that $O(n)$ has exactly 2 connected components.

Problem 2.

Let X and Y be locally compact, Hausdorff topological spaces. A continuous function $f : X \to Y$ is called *proper* if $f^{-1}K$ is compact whenever $K \subset Y$ is compact. It is called closed if $f(Z)$ is closed in Y whenever Z is closed in X. Show that if f is proper then f is closed.

Problem 3. Let X and Y be connected semilocally simply connected topological spaces and let $X \vee Y$ be the topological space obtianed by gluing X and Y at a point e. Let E be the universal cover of $X \vee Y$. Assume that the restriction of E to X is connected. Prove that Y is simply connected.

Problem 4.

Let M be a smooth 2-dimensional manifold, and let (U, ϕ), (U, ψ) be two smooth charts on M with the same domain. Assume that the change of coordinates $\psi \circ \phi^{-1}$ is given by the formula

$$(x, y) = (\psi \circ \phi^{-1})(u, v) = (u \cos v, u \sin v),$$

and that $\psi \circ \phi^{-1}$ maps the region

$$\{(u, v) \in \mathbb{R}^2 \mid u > 0,\ 0 < v < \pi\}$$

onto the upper half plane

$$\{(x, y) \in \mathbb{R}^2 \mid y > 0\}.$$

(a) If a 1-form η on M has the local coordinate expression

$$\eta = y\, dx$$

in the chart (U, ψ), find the local coordinate expression for η in the chart (U, ϕ).

(b) If a vector field X on M has the local coordinate expression

$$X = \frac{\partial}{\partial v}$$

in the chart (U, ϕ), find the local coordinate expression for X in the chart (U, ψ).

Problem 5. Let $M_{2\times 2}(\mathbb{R})$ be the space of 2×2 matrices with real entries, let $S_{2\times 2}(\mathbb{R})$ be the space of *symmetric* 2×2 matrices with real entries, and let $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Define a map $f : M_{2\times 2}(\mathbb{R}) \to S_{2\times 2}(\mathbb{R})$ by

$$f(A) = A^T J A.$$

(a) Compute f and the tangent map Df explicitly in terms of coordinates. (Use the standard identifications $M_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^4$ and $S_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^3$ to define coordinates on each space, so that f can be regarded as a map from $\mathbb{R}^4$ to $\mathbb{R}^3$.)

(b) Show that if A is invertible, then the tangent map

$$Df\big|_A : T_A(M_{2\times 2}(\mathbb{R})) \to T_{f(A)}(S_{2\times 2}(\mathbb{R}))$$

has maximum rank.

(c) Show that the set

$$\{A \in M_{2\times 2}(\mathbb{R}) \mid A^T J A = J\}$$

is a smooth submanifold of $M_{2\times 2}(\mathbb{R})$.

Problem 6.

Define a 1-form ω on $\mathbb{R}^2 \setminus \{(0,0)\}$ by

$$\omega = -\left(\frac{y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy.$$

(a) Let C be the circle of radius $r > 0$ centered at the origin, oriented counterclockwise. Evaluate the integral $\int_C \omega$ by direct computation.

(b) Calculate $d\omega$.

(c) Let C' be the curve defined implicitly by the equation $x^4 + y^2 = 1$, oriented counterclockwise. Compute the integral $\int_{C'} \omega$. (Hint: This should not require any explicit computation!)