RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Topology-Geometry

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

August, 2023

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1.

(a) Suppose that X is compact and Hausdorff. Prove that X is regular.

(b) Suppose that Y is locally compact and Hausdorff. Prove that Y is regular.

Problem 2.

(a) Let Y be a path connected topological space and view Y as a subspace of $\mathbb{R}^2 \vee Y$ via inclusion. Prove that Y is deformation retract of $\mathbb{R}^2 \vee Y$.

(Here we form the wedge sum over a point $y_0 \in Y$ and the origin in $\mathbb{R}^2$. Therefore, by definition, $\mathbb{R}^2 \vee Y$ is the disjoint union of $\mathbb{R}^2$ and Y modulo the relation $y_0 \in Y \sim (0,0) \in \mathbb{R}^2$).

(b) With Y and y_0 as in the previous question, compute $\pi_1(\mathbb{R}^2 \vee Y, y_0)$ in terms of $\pi_1(Y, y_0)$.

(c) Recall that a surface, X, is a second countable, Hausdorff space with the property that for each $x \in X$ there exists an open neighborhood of x, U, and a homeomorphism $h : U \to \mathbb{R}^2$. You can in addition assume that $h(x) = (0,0) \in \mathbb{R}^2$.

Prove the following: if N and M are path connected surfaces, then

$$\pi_1(N \vee M, y_0) \cong \pi_1(N, y_0) * \pi_1(M, y_0)$$

where y_0 is the point the wedge product is over.

Problem 3.

(a) Suppose $p : E \to B$ is a covering map with E path connected and B simply connected. Prove that p is a homeomorphism.

(b) Show that if the universal cover of a space exists, then it is unique (up to homeomorphism).

(c) For which $n \in \mathbb{N}$, does there exists a covering map from $\mathbb{R}^n$ to $\mathbb{R}P^n$ (here $\mathbb{R}P^n$ denotes real projective space)? You must justify your answer.

Problem 4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x,y,z) = (x,y)$, be the projection onto the first two factors, and consider the following two vector fields on $\mathbb{R}^3$:

$$\chi = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2(y+1)^2 z\frac{\partial}{\partial z} \qquad \psi = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + 4(y+1)^2 z\frac{\partial}{\partial z}$$

(a) At what points $p \in \mathbb{R}^3$ do $(T_p f)(\chi(p))$ and $(T_p f)(\psi(p))$ span $T_{f(p)}\mathbb{R}^2$, where $T_p f$ denotes the differential of f at p.

(b) Consider the path $\gamma : [0,1] \to \mathbb{R}^2$, $\gamma(t) = (t+2, t)$. Find all lifts $\tilde{\gamma}$ of γ to $\mathbb{R}^3$ (i.e., $\tilde{\gamma} : [0,1] \to \mathbb{R}^3$ is a C^∞ map and $f \circ \tilde{\gamma} = \gamma$) such that $\tilde{\gamma}(0) = (2,0,1)$ and $\tilde{\gamma}'(t)$ is in the span of $\chi(\tilde{\gamma}(t))$ and $\psi(\tilde{\gamma}(t))$ for all $t \in [0,1]$.

Problem 5. Let ω be a closed n-form on $\mathbb{R}^{n+1} - \{(0,\ldots,0)\}$, let $S^n \subseteq \mathbb{R}^{n+1}$ be the unit n-sphere, and let $X^\circ = \{(x,y,z) \in \mathbb{R}^3 - \{(0,0,0)\} \mid x^2 + y^2 - z^2 = 0\}$.

(a) Show that if ω is exact, then $\int_{S^n} \omega = 0$.

(b) Show the converse of (a) for $n = 1$. [*Hint:* integrate ω along paths.]

(c) Show that X° is an embedded submanifold of $\mathbb{R}^3$, and find an explicit basis for the vector space of closed 1-forms on X° modulo the vector space of exact 1-forms on X°. [*Hint*: consider $\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ on $\mathbb{R}^2 - \{(0,0)\}$.]

Problem 6. Let $f : X \to Y$ be a C^∞ map of smooth connected manifolds of the same dimension n, and let $U \subseteq X$ be a dense open subset. Suppose there exist a Lie group G of $n \times n$ matrices, as well as local coordinate charts for X and Y so that with respect to these coordinate charts the differential of f at each point $p \in U$ lies in G.

(a) Show that f is a local diffeomorphism at every point $p \in U$ (i.e., for every point $p \in U$ there is an open neighborhood $V \subseteq X$ of p such that $f|_V : V \to f(V)$ is a diffeomorphism).

(b) Show by example that f need not be a local diffeomorphism at all points $p \in X$.

(c) Show that if $G = \mathrm{O}(n)$, the orthogonal group, then f is locally a diffeomorphism at all points $p \in X$ (recall that a square matrix A is orthogonal if $AA^T = A^T A = I$, where I is the identity matrix).