RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

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INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Q.1 Let X be a topological space, let Y be a set, and let $f : X \to Y$ be a function. Construct a topology on Y with the following property: if Z is a topological space and $g : Y \to Z$ is a function, then g is continuous if and only if $g \circ f$ is continuous. Prove that your topology has the required property.

Q.2 Prove that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic unless $n = m$.

Q.3 Recall that the n-th homotopy group $\pi_n(X, x)$ of a topological space X with a basepoint x is the set of basepoint preserving homotopy classes of maps $S^n \to X$ that send the basepoint of S^n to x. Suppose that $f : X \to Y$ is a covering space. Prove that the map $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$, sending $\alpha : S^n \to X$ to $f \circ \alpha$, is a bijection for all $n \geq 2$. (Note that the group structure on $\pi_n(X, x)$ is not relevant to this problem.)

Q.4 Let $a, b \in \mathbb{R}$, and consider the subset S of $\mathbb{R}^3$ defined by the equations

$$xyz = a, \qquad x + y + z = b.$$

(a) Show that if $a \neq 0$ and $b^3 \neq 27a$, then S is a smooth submanifold of $\mathbb{R}^3$.

(b) Suppose that $a = 0$ and $b = 1$. Identify the points of S where S is not a smooth submanifold of $\mathbb{R}^3$.

Q.5 Consider the two vector fields on $\mathbb{R}^3$ with coordinates (x, y, z) given by

$$X = \frac{\partial}{\partial y} + z\frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial z} + y\frac{\partial}{\partial x}.$$

(a) Show that $[X, Y] = 0$.

(b) Compute the flows θ_t of X and ϕ_s of Y, and show directly that for any point $p = (a, b, c) \in \mathbb{R}^3$ and any $s, t \in \mathbb{R}$,

$$\theta_t(\phi_s(p)) = \phi_s(\theta_t(p)).$$

(c) Use part (b) to give a parametrization $(x(s,t), y(s,t), z(s,t))$ for the (unique!) surface passing through the point $p = (1, 0, 0)$ and tangent to the vector fields X and Y at each point. Then give an equation of the form $F(x, y, z) = 0$ that describes this surface.

Q.6 Define a 1-form ω on $\mathbb{R}^2 \setminus \{(0,0)\}$ by

$$\omega = -\left(\frac{y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy.$$

(a) Let C be the circle of radius $r > 0$ centered at the origin, oriented counterclockwise. Evaluate the integral $\int_C \omega$ by direct computation.

(b) Calculate $d\omega$.

(c) Let C' be the curve defined implicitly by the equation $x^4 + y^2 = 1$, oriented counterclockwise. Compute the integral $\int_{C'} \omega$.